Transverse instability of solitary waves in Korteweg fluids

Sylvie Benzoni-Gavage

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Abstract. The Euler-Korteweg model is made of the standard Euler equations for compressible fluids supplemented with the Korteweg tensor, which is intended to take into account capillary effects. For non-monotone 'pressure' laws, the Euler-Korteweg model is known to admit solitary waves, even though their physical significance remains unclear. In fact, several kinds of solitary waves, with various endstates, can be identified. In one space dimension, all these solitary waves may be viewed as critical points under constraint of the total energy, the constraint being linked to translational invariance. In an earlier work with Danchin, Descombes and Jamet [Interf. Free Bound. 2005], a sufficient condition was obtained for their orbital stability, by the method of Grillakis, Shatah and Strauss [Journal of Functional Analysis, 1987], relying on the Hamiltonian structure and on the translational invariance. Numerical evidence was given that this condition is satisfied by some dynamic solitary waves, whereas it fails for solitary waves closer to thermodynamic equilibrium. That condition is of the form $m''(\sigma) > 0$, with σ the speed and m the constrained energy of the wave. It turns out that, as was already known in other contexts, $m''(\sigma)$ is linked to the low frequency behavior of the Evans function associated with the linearized equations. This link was investigated by Zumbrun [Z. Anal. Anwend. 2008] (and independently by Bridges and Derks) for simplified equations (with constant capillarity) in Lagrangian coordinates. Zumbrun proved in that context that $m''(\sigma) \geq 0$ is necessary for linearized stability. This result is revisited here with general capillarities in Eulerian coordinates, and the main purpose is to investigate the *multidimensional* stability of planar solitary waves. In this respect, variational tools are not much appropriate. Nevertheless, the Evans function technique does extend to arbitrary space dimensions, and its low-frequency behavior can be computed explicitly. It turns out from this behavior and an argument pointed out by Zumbrun and Serre [Indiana Univ. Math. J 1999] that planar solitary wave solutions of the Euler-Korteweg model are linearly unstable with respect to transverse perturbations of large wave length.

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1 Introduction

We consider a fluid whose free energy is allowed to depend on the gradient of density in the following way

$$F(\rho, \nabla\rho) \,=\, F_0(\rho) \,+\, \frac{1}{2}\, K(\rho)\, |\nabla\rho|^2 \,.$$

Here above, $K(\rho)$ stands for a capillarity coefficient depending on ρ, and is supposed to be positive for all positive values of ρ. If dissipation phenomena are neglected, the corresponding isothermal equations

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of motion – which can be found by classical principles of mechanics [14, 19] – are

$$\text{(1.1)}\qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \nabla (\rho \operatorname{div}(K \nabla\rho)) - \operatorname{div}(K \nabla\rho \otimes \nabla\rho), \end{cases}$$

where $p := \rho \frac{\partial F}{\partial \rho} - F$ also depends on $\nabla\rho$. By definition,

$$p(\rho, \nabla\rho) = p_0(\rho) + \frac{1}{2}(\rho K'(\rho) - K(\rho)) |\nabla\rho|^2, \quad p_0 := \rho \frac{\partial F_0}{\partial \rho} - F_0.$$

For smooth solutions, (1.1) is easily seen to be equivalent to

$$\text{(1.2)}\qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla g_0 = \nabla(K \Delta\rho + \frac{1}{2} K'_\rho |\nabla\rho|^2), \end{cases}$$

where g_0 is the standard chemical potential of the fluid, defined by

$$g_0 = \frac{\mathrm{d}F_0}{\mathrm{d}\rho},$$

and such that

$$\frac{\mathrm{d}g_0}{\mathrm{d}\rho} = \frac{1}{\rho} \frac{\mathrm{d}p_0}{\mathrm{d}\rho}.$$

In one space dimension, (1.2) reduces to

$$\text{(1.3)}\qquad \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \\ \partial_t u + u \partial_x u + \partial_x(g_0) = \partial_x(K \partial^2_{xx}\rho + \frac{1}{2} K'_\rho (\partial_x\rho)^2), \end{cases}$$

which admits the formal Hamiltonian formulation

$$\text{(1.4)}\qquad \partial_t \mathbf{U} = \mathcal{J} \delta\mathcal{H}[\mathbf{U}]$$

where

$$\mathbf{U} := \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix},$$

$$\mathcal{H}[\mathbf{U}] := \int H(\mathbf{U}, \partial_x \mathbf{U}) \, \mathrm{d}x, \quad H(\mathbf{U}, \partial_x \mathbf{U}) = \frac{1}{2} \rho u^2 + F_0(\rho) + \frac{1}{2} K(\rho) (\partial_x \rho)^2,$$

and

$$\delta\mathcal{H}[\mathbf{U}] = \begin{pmatrix} \frac{1}{2} u^2 + g_0(\rho) - K(\rho) \partial_x^2 \rho - \frac{1}{2} \frac{\mathrm{d}K}{\mathrm{d}\rho}(\rho) (\partial_x \rho)^2 \\ \rho u \end{pmatrix}.$$

To make this formulation correct we may prescribe the behavior of $\mathbf{U}$ at infinity, and change the integral of H accordingly, in order to turn it into a convergent one. As far as perturbations of solitary waves are concerned, we may assume that $\mathbf{U}$ converges (exponentially fast) to some limit $\mathbf{U}_\infty$ at $\pm\infty$. Then

$$\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty] := \int \Big(H(\mathbf{U}, \partial_x \mathbf{U}) - H(\mathbf{U}_\infty, 0) - \delta\mathcal{H}[\mathbf{U}_\infty] \cdot (\mathbf{U} - \mathbf{U}_\infty) \Big) \, \mathrm{d}x$$

is well defined for $\mathbf{U} \in \mathbf{U}_\infty + (H^1 \times L^2)$, and for such $\mathbf{U}$, (1.3) equivalently reads

$$\text{(1.5)}\qquad \partial_t \mathbf{U} = \mathcal{J} \delta\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty].$$

Here above, the notation δ stands for the variational gradient with respect to $\mathbf{U}$, the endstate $\mathbf{U}_\infty$ being kept fixed. A solitary wave is by definition a homoclinic traveling wave solution, that is, a solution that propagates a same profile, say $\underline{\mathbf{U}}$, at constant speed, say σ, with a same endstate $\mathbf{U}_\infty$ at $+\infty$ and $-\infty$. For a nonmonotone pressure law $p_0 = p_0(\rho)$, or equivalently, for a nonconvex free energy $F_0 = F_0(\rho)$, (1.3) is known to admit solitary waves, that is, global smooth solutions of the form

$$\mathbf{U}(x,t) \,=\, \underline{\mathbf{U}}(x-\sigma t)\,, \quad \lim_{\xi\to\pm\infty} \underline{\mathbf{U}}(\xi) \,=\, \mathbf{U}_\infty\,.$$

The existence of solitary waves follows from a simple phase portrait analysis of the governing ODEs, which appear to be Hamiltonian too (a general fact, see [2] p. 11–12), see [6] for more details. Solitary waves – unlike heteroclinic connections – persist under perturbation of the speed σ. Moreover, solitary waves can be viewed, in one space dimension, as critical points of the Hamiltonian $\widetilde{\mathcal{H}}$ under the constraint

$$\widetilde{\mathcal{Q}}[\mathbf{U};\mathbf{U}_\infty] \,:=\, \int \Big((\rho-\rho_\infty)\,(u-u_\infty) \Big)\,\mathrm{d}x\,.$$

Indeed, working in the abstract Hamiltonian setting described above, we may write the traveling wave ODEs as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(-\sigma\,\underline{\mathbf{U}} \,+\, \mathbf{J}\,\delta\widetilde{\mathcal{H}}[\underline{\mathbf{U}};\mathbf{U}_\infty] \Big) \,=\, 0\,, \qquad \mathbf{J} \,:=\, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \xi \,=\, x \,-\, \sigma\,t\,,$$

hence, multiplying the ODE by $\mathbf{J}$ and using that $\mathbf{J}^2 \,=\, \mathbf{I}$,

$$\delta\widetilde{\mathcal{H}}[\underline{\mathbf{U}};\mathbf{U}_\infty] \,-\, \sigma\,\mathbf{J}\,\underline{\mathbf{U}} \,\equiv\, \text{constant}\,.$$

Evaluating at $\pm\infty$, we see that the constant must be $-\,\sigma\,\mathbf{J}\,\mathbf{U}_\infty$, and since $\mathbf{J}\,(\underline{\mathbf{U}} - \mathbf{U}_\infty) \,=\, \delta\widetilde{\mathcal{Q}}[\underline{\mathbf{U}};\mathbf{U}_\infty]$, we obtain

$$\delta(\widetilde{\mathcal{H}} - \sigma\widetilde{\mathcal{Q}})[\underline{\mathbf{U}};\mathbf{U}_\infty] \,\equiv\, \mathbf{0}\,. \tag{1.6}$$

As claimed above, this means that $\underline{\mathbf{U}}$ is a critical point of $\widetilde{\mathcal{H}}$ under the constraint $\widetilde{\mathcal{Q}}$, with associated Lagrange multiplier σ (the speed of the wave). The fact that $\widetilde{\mathcal{Q}}$ is a conserved quantity along solutions of (1.3) (in $\mathbf{U}_\infty \,+\, \mathscr{C}^1(\mathbb{R};H^1 \times L^2)$) is linked to translational invariance. Indeed, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{Q}}[\mathbf{U};\mathbf{U}_\infty] \,=\, \int \Big(\delta\widetilde{\mathcal{Q}}[\mathbf{U};\mathbf{U}_\infty]\cdot\partial_t\mathbf{U} \Big)\,\mathrm{d}x \,=\, -\int \Big(\mathbf{J}(\mathbf{U}-\mathbf{U}_\infty)\cdot\mathbf{J}\partial_x\delta\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty] \Big)\,\mathrm{d}x$$

$$= \int \Big(\delta\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty]\cdot\partial_x\mathbf{U} \Big)\,\mathrm{d}x$$

after integration by parts (and using that $\mathbf{J}^t\mathbf{J} \,=\, \mathbf{I}$), and the nullity of the last integral follows from the equality

$$\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{H}}[\mathbf{U}_s;\mathbf{U}_\infty] \,=\, 0$$

for $\mathbf{U}_s(x,t) := \mathbf{U}(x+s,t)$. This very same translational invariance also implies that solitary waves of given speed σ and endstate $\mathbf{U}_\infty$, form a one-parameter family $(\underline{\mathbf{U}}_s)_{s\in\mathbb{R}}$, with $\underline{\mathbf{U}}(\xi) \,=\, \underline{\mathbf{U}}(\xi+s)$. In addition, we see on (1.6) that

$$\widetilde{\mathcal{H}}[\underline{\mathbf{U}}_s;\mathbf{U}_\infty] \,-\, \sigma\,\widetilde{\mathcal{Q}}[\underline{\mathbf{U}}_s;\mathbf{U}_\infty]$$

does not depend on s. So there is no ambiguity in defining

$$m(\sigma;\mathbf{U}_\infty) \,:=\, \widetilde{\mathcal{H}}[\underline{\mathbf{U}};\mathbf{U}_\infty] \,-\, \sigma\,\widetilde{\mathcal{Q}}[\underline{\mathbf{U}};\mathbf{U}_\infty]\,.$$

This constrained energy plays a crucial role in the one-dimensional stability analysis of the wave $\underline{\mathbf{U}}$. As observed in [6], the actual computation of $m(\sigma;\mathbf{U}_\infty)$ does not require the resolution of the traveling wave ODEs, and can be done in the phase plane. Indeed, the special form of

$$\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty] \,=\, \int \Big(H_0(\mathbf{U};\mathbf{U}_\infty) \,+\, \frac{1}{2}\,K(\rho)\,(\partial_x\rho)^2 \Big)\,\mathrm{d}x$$

implies that

$$\delta\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty]\cdot\partial_x\mathbf{U} \,=\, \partial_x \Big(H_0(\mathbf{U};\mathbf{U}_\infty) \,-\, \frac{1}{2}\,K(\rho)\,(\partial_x\rho)^2 \Big)\,,$$

so that $\mathrm{d}\underline{\mathbf{U}}/\mathrm{d}\xi$ is an integrating factor of (1.6). The integrated equation reads

$$H_0(\underline{\mathbf{U}};\mathbf{U}_\infty) \,-\, \sigma\,(\rho-\rho_\infty)\,(\underline{u}-u_\infty) \,-\, \frac{1}{2}\,K(\rho)\,\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2 \,\equiv\, 0\,,$$

hence

$$m(\sigma;\mathbf{U}_\infty) \,=\, \int K(\rho)\,\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2\,\mathrm{d}\xi \,=\, 2\int_{\xi_0}^{+\infty} K(\rho)\,\left(\frac{\mathrm{d}\rho}{\mathrm{d}\xi}\right)^2\,\mathrm{d}\xi\,,$$

where ξ_0 is the center of symmetry of the soliton. To compute $m(\sigma;\mathbf{U}_\infty)$ in the phase plane it suffices to make the change of variables $r=\rho(\xi)$ for $\xi\in(\xi_0,+\infty)$ and use the formula

$$\frac{\mathrm{d}\rho}{\mathrm{d}\xi} \,=\, \pm\left(\frac{2}{K(\rho)}\,\big(H_0(\underline{\mathbf{U}};\mathbf{U}_\infty) \,-\, \sigma\,(\rho-\rho_\infty)\,(\underline{u}-u_\infty)\big)\right)^{1/2}\,.$$

2 One dimensional stability criterion

In what follows we omit the tilda on $\mathcal{H}$ and $\mathcal{Q}$ for simplicity, and we emphasize with a superscript the dependence on σ of solitary waves.

Theorem 1 *We fix an endstate $\mathbf{U}_\infty$, and assume that, for all σ in an open interval there exists a solitary wave solution of* (1.3), *$\underline{\mathbf{U}}^\sigma$, of speed σ and endstate $\mathbf{U}_\infty$. We consider the function m defined by*

$$m(\sigma;\mathbf{U}_\infty) \,:=\, \mathcal{H}[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty] \,-\, \sigma\,\mathcal{Q}[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty]\,,$$

the functionals $\mathcal{H}$ and $\mathcal{Q}$ being defined by

$$\mathcal{H}[\mathbf{U};\mathbf{U}_\infty] \,:=\, \int \Big(H_0(\mathbf{U}) \,-\, H_0(\mathbf{U}_\infty) \,+\, \frac{1}{2}\,K(\rho)\,(\partial_x\rho)^2 \\ -\,\partial_\rho H_0(\mathbf{U}_\infty)\,(\rho-\rho_\infty) \,-\, \partial_u H_0(\mathbf{U}_\infty)\,(u-u_\infty) \Big)\,\mathrm{d}x$$

with

$$H_0(\rho,u) \,:=\, \frac{1}{2}\,\rho\,u^2 \,+\, F_0(\rho)\,,$$

and

$$\mathcal{Q}[\mathbf{U};\mathbf{U}_\infty] \,:=\, \int \Big((\rho-\rho_\infty)\,(u-u_\infty) \Big)\,\mathrm{d}x\,.$$

- *The solitary wave $\underline{\mathbf{U}}^\sigma$ is* orbitally stable *if*

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma;\mathbf{U}_\infty) \,>\, 0\,.$$

- *It is* linearly unstable *if*

$$\frac{\partial^2 m}{\partial \sigma^2}(\sigma; \mathbf{U}_\infty) \,<\, 0\,.$$

Remark 1 *As mentioned before, solitary waves can be found by phase portrait analysis. For double-well free energy, typical of van der Waals fluids, this matter is investigated in details in [6], with a classification of solitary waves according to their endstate (liquid or vapor) and their amplitude.*

Proof. [Theorem 1] The sufficient condition $m''(\sigma) > 0$ for orbital stability can be deduced from the abstract result of Grillakis, Shatah and Strauss [12]: this was already pointed out by Bona and Sachs in [8] for the 'good' Boussinesq equation, a special case of (1.3) rewritten in Lagrangian coordinates; for the general system (1.3), see [6]. That $m''(\sigma) < 0$ implies instability cannot be deduced from the Grillakis-Shatah-Strauss result – which is an if-and-only-if result for orbital stability –, basically because the operator $\mathcal{J}$ is not onto. However, an Evans function calculation does yield a necessary condition for stability, as was shown by Zumbrun [21] in a Lagrangian framework (also see [9]) with a constant capillarity coefficient κ, related to the Eulerian capillarity coefficient by $\kappa = K\rho^5$. We are going to perform this calculation in the Eulerian framework with an arbitrary capillarity coefficient K. We first make standard observations on the profile equation

$$(\delta\mathcal{H} \,-\, \sigma\,\delta\mathcal{Q})[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \,\equiv\, 0 \tag{2.7}$$

(which is just (1.6) with slightly different notations). The variational form of (2.7) has two crucial consequences regarding the second-order differential operator

$$\mathcal{L}^\sigma \,:=\, (\,\mathrm{Hess}\mathcal{H} \,-\, \sigma\,\mathrm{Hess}\mathcal{Q}\,)[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty]\,.$$

The first consequence is linked to translational invariance. Indeed, all translated profiles $\underline{\mathbf{U}}^\sigma_s \,:\, \xi \,\mapsto\, \underline{\mathbf{U}}^\sigma(\xi + s)$ satisfy the same equation (2.7). Therefore, differentiating

$$(\delta\mathcal{H} \,-\, \sigma\,\delta\mathcal{Q})[\underline{\mathbf{U}}^\sigma_s; \mathbf{U}_\infty] \,\equiv\, 0$$

with respect to s and evaluating at $s = 0$ we find that $\partial_\xi \underline{\mathbf{U}}^\sigma$ is in the kernel of $\mathcal{L}^\sigma$. The second consequence is obtained by differentiating (2.7) with respect to σ. This yields

$$\mathcal{L}^\sigma \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,=\, \delta\mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty]\,. \tag{2.8}$$

To address the linearized stability of $\underline{\mathbf{U}}^\sigma$, the first, usual step consists in making a change of Galilean frame $(x,t) \,\mapsto\, (\xi := x - \sigma t, t)$, so as to make the wave stationary. This clearly changes the abstract form of (1.3),

$$\partial_t \mathbf{U} \,=\, -\partial_x \mathbf{J}\,\delta\mathcal{H}[\mathbf{U}; \mathbf{U}_\infty]\,,$$

into

$$\partial_t \mathbf{U} \,-\, \sigma\,\partial_\xi \mathbf{U} \,=\, -\partial_\xi \mathbf{J}\,\delta\mathcal{H}[\mathbf{U}; \mathbf{U}_\infty]\,.$$

Linearizing about $\underline{\mathbf{U}}^\sigma$ we are led to

$$\partial_t \dot{\mathbf{U}} \,-\, \sigma\,\partial_\xi \dot{\mathbf{U}} \,=\, -\partial_\xi \mathbf{J}\,(\mathrm{Hess}\mathcal{H})[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \cdot \dot{\mathbf{U}}\,,$$

or equivalently, observing that $\dot{\mathbf{U}} \,=\, \mathbf{J}^2 \dot{\mathbf{U}} \,=\, \mathbf{J}\,(\mathrm{Hess}\mathcal{Q})[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \cdot \dot{\mathbf{U}}$,

$$\partial_t \dot{\mathbf{U}} \,\;=\, -\partial_\xi \mathbf{J}\,\mathcal{L}^\sigma \cdot \dot{\mathbf{U}}\,.$$

Introducing the third-order differential operator $L^\sigma \,:=\, -\partial_\xi \mathbf{J}\,\mathcal{L}^\sigma$, we infer from (2.8) that $L^\sigma \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,=\, -\,\partial_\xi \mathbf{J}\,\delta\mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty]$, that is,

$$L^\sigma \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,=\, -\,\partial_\xi \underline{\mathbf{U}}^\sigma\,. \tag{2.9}$$

Since

$$L^\sigma \cdot \partial_\xi \underline{\mathbf{U}}^\sigma \;=\; -\,\partial_\xi \mathbf{J}\,\mathcal{L}^\sigma \cdot \partial_\xi \underline{\mathbf{U}}^\sigma \;=\; 0\,,$$

this means that 0 is an eigenvalue of L^σ of algebraic multiplicity greater or equal to 2. It will turn out that, if

$$\frac{\partial^2 m}{\partial \sigma^2}(\sigma; \mathbf{U}_\infty) \;\neq\; 0\,,$$

the eigenvalue 0 is exactly of multiplicity 2, or equivalently, the Evans function associated to L^σ has a zero of multiplicity two at zero. This will follow from Lemma 1 below and the more explicit formula

$$\frac{\partial^2 m}{\partial \sigma^2}(\sigma; \mathbf{U}_\infty) \;=\; -\int \Big((\underline{\rho}^\sigma - \rho_\infty)\, \partial_\sigma \underline{u}^\sigma \;+\; (\underline{u}^\sigma - u_\infty)\, \partial_\sigma \underline{\rho}^\sigma \Big)\, \mathrm{d}\xi\,. \tag{2.10}$$

The latter comes from the definition of m, which implies

$$\frac{\partial m}{\partial \sigma}(\sigma; \mathbf{U}_\infty) \;=\; \int (\delta \mathcal{H} \,-\, \sigma\, \delta \mathcal{Q})[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,\mathrm{d}\xi \;-\; \mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \;=\; -\,\mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty]$$

because of (2.7), hence

$$\frac{\partial^2 m}{\partial \sigma^2}(\sigma; \mathbf{U}_\infty) \;=\; -\int \delta \mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,\mathrm{d}\xi\,. \tag{2.11}$$

Lemma 1 below shows that $\partial^2 m / \partial \sigma^2$ is proportional to the second-order derivative of the Evans function. More precisely, if $\partial^2 m / \partial \sigma^2$ is negative, then the Evans function changes sign in between 0 and $+\infty$, so that by the mean value theorem it must vanish at some positive λ, which is therefore an unstable eigenvalue of the linear operator L^σ. □

Remark 2 *The profile $\underline{\mathbf{U}}^\sigma$ is a critical point of the constrained functional $\mathcal{H} - \sigma \mathcal{Q}$, and the Hessian at $\underline{\mathbf{U}}^\sigma$ of that functional is precisely*

$$\mathcal{L}^\sigma \;=\; \begin{pmatrix} \mathcal{M}_0 & \underline{u}^\sigma - \sigma \\ \underline{u}^\sigma - \sigma & \underline{\rho}^\sigma \end{pmatrix}, \quad \mathcal{M}_0 \;:=\; -\,\partial_\xi \underline{K}^\sigma \partial_\xi \,+\, \underline{\alpha}^\sigma\,.$$

The operator $\mathcal{L}^\sigma$ is not *monotone if $\underline{\mathbf{U}}^\sigma$ is homoclinic. It would be monotone if the Sturm-Liouville operator*

$$\mathcal{M} \;:=\; \mathcal{M}_0 \,-\, \frac{1}{2}\,(\underline{u}^\sigma - \sigma)^2$$

were so. But, $\mathcal{L}^\sigma \cdot \partial_\xi \underline{\mathbf{U}}^\sigma \;=\; 0$ implies that $\partial_\xi \underline{\rho}^\sigma$ is in the kernel of $\mathcal{M}$, and since $\partial_\xi \underline{\rho}^\sigma$ vanishes (once), 0 is the second *eigenvalue of $\mathcal{M}$. In fact, this implies that 0 is also the second eigenvalue of $\mathcal{L}^\sigma$ (see Appendix B in [6] for details). Note in addition that by* (2.8) *and* (2.11)*,*

$$\frac{\partial^2 m}{\partial \sigma^2}(\sigma; \mathbf{U}_\infty) \;=\; -\,\langle \mathcal{L}^\sigma \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,,\, \partial_\sigma \underline{\mathbf{U}}^\sigma \rangle_{L^2}\,.$$

Hence the stable case $\partial^2 m / \partial \sigma^2 > 0$ corresponds to when

$$\langle \mathcal{L}^\sigma \cdot \partial_\sigma \underline{\mathbf{U}}^\sigma \,,\, \partial_\sigma \underline{\mathbf{U}}^\sigma \rangle_{L^2} \;<\; 0\,.$$

The main result in [12] shows that this 'bad' direction $\partial_\sigma \underline{\mathbf{U}}^\sigma$ can then be factored out, in that

$$\langle \mathcal{L}^\sigma \cdot \mathbf{Y} \,,\, \mathbf{Y} \rangle_{L^2} \;\geq\; 0 \qquad \text{for all } \mathbf{Y} \text{ such that } \quad \langle \delta \mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \,,\, \mathbf{Y} \rangle_{L^2} \;=\; 0\,.$$

Lemma 1 *If* (2.7) *admits a homoclinic solution then the endstate is necessarily subsonic, that is,*

$$\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) \,>\, (u_\infty - \sigma)^2\,, \tag{2.12}$$

and the essential spectrum of the linear operator

$$L^\sigma \;=\; -\partial_\xi \mathbf{J}\,(\,\mathit{Hess}\mathcal{H} \,-\, \sigma\,\mathit{Hess}\mathcal{Q}\,)[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty]$$

consists of the imaginary axis. Furthermore, L^σ can be associated with a smooth Evans function D^σ : $\lambda \in [0,+\infty) \to \mathbb{R}$, such that

$$\forall \lambda > 0\,,\,(\,D^\sigma(\lambda) = 0 \quad\Leftrightarrow\quad \mathit{Ker}(L^\sigma - \lambda) \,\neq\, \{0\}\,)\,,$$

and $D^\sigma(0) = 0$, $(D^\sigma)'(0) = 0$, $D^\sigma(\lambda) > 0$ for $\lambda \gg 1$,

$$\mathit{sgn}(D^\sigma)''(0) \;=\; -\,\mathit{sgn}\int \Big((\underline{\rho}^\sigma - \rho_\infty)\,\partial_\sigma \underline{u}^\sigma \;+\; (\underline{u}^\sigma - u_\infty)\,\partial_\sigma \underline{\rho}^\sigma \Big)\,\mathrm{d}\xi\,.$$

Proof. The profile equation (2.7) can be rewritten more explicitly as

$$\left\{\begin{array}{l} \underline{\rho}^\sigma\,(\underline{u}^\sigma \,-\, \sigma) \;\equiv\; \rho_\infty\,(u_\infty \,-\, \sigma)\,, \\ \\ K(\underline{\rho}^\sigma)\partial^2_{\xi\xi}\underline{\rho}^\sigma + \frac{1}{2}\partial_\xi K(\underline{\rho}^\sigma)\,\partial_\xi\underline{\rho}^\sigma - g_0(\underline{\rho}^\sigma) + g_0(\rho_\infty) - \frac{1}{2}(\underline{u}^\sigma - \sigma)^2 + \frac{1}{2}(u_\infty - \sigma)^2 = 0\,. \end{array}\right. \tag{2.13}$$

• **Subsonicity of the enstate.** We may eliminate the velocity $\underline{u}^\sigma$ from (2.13) and rewrite the second equation (of second order) as the planar system

$$\left\{\begin{array}{l} \phi' \;=\; \frac{1}{\sqrt{K(\phi)}}\,\psi\,, \\ \psi' \;=\; \frac{1}{\sqrt{K(\phi)}}\,\Big(g_0(\phi) \,+\, \frac{1}{2}\,\frac{j^2}{\phi^2} \,-\, \mu\Big)\,, \end{array}\right. \tag{2.14}$$

with the simplifying notations $\phi := \underline{\rho}^\sigma$, $j := \rho_\infty\,(u_\infty \,-\, \sigma)$, and $\mu := g_0(\rho_\infty) \,+\, \frac{1}{2}\,\frac{j^2}{\rho_\infty^2}$. The matrix of the linearized system at $(\rho_\infty, 0)$ is

$$\frac{1}{\sqrt{K(\phi)}}\begin{pmatrix} 0 & 1 \\ \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) \,-\, \frac{j^2}{\rho_\infty^3} & 0 \end{pmatrix},$$

which is hyperbolic if and only if

$$\frac{1}{\rho_\infty}\,\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) \;=\; \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) \;>\; \frac{j^2}{\rho_\infty^3} \;=\; \frac{(u_\infty - \sigma)^2}{\rho_\infty}\,.$$

In other words, the fixed point $(\rho_\infty, 0)$ of (2.14) is a saddle-point if (2.12) holds true, and a center if $\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) \,<\, (u_\infty - \sigma)^2$. For a homoclinic connection to exist, $(\rho_\infty, 0)$ must be a saddle-point, hence the necessary condition (2.12). Note that (2.12) implies in particular

$$\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) \,>\, 0\,,$$

which means that the density ρ_∞ corresponds to a thermodynamically stable state, where we have a real sound speed

$$c_\infty \;:=\; \sqrt{\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty)}\,.$$

(Recall that the existence and classification of solitary waves has been discussed in [6, 7] .)

• **Essential spectrum of the linearized operator.** Regarding the essential spectrum of L^σ, we have to concentrate on the asymptotic operator L^σ_∞, obtained by freezing the coefficients at $\pm\infty$,

$$L^\sigma_\infty \cdot \dot{\mathbf{U}} \;:=\; \begin{pmatrix} -\,(u_\infty-\sigma)\,\partial_\xi\dot\rho \;-\; \rho_\infty\,\partial_\xi\dot u \\ -\,(u_\infty-\sigma)\,\partial_\xi\dot u \;-\; \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)\,\partial_\xi\dot\rho \;+\; K(\rho_\infty)\,\partial^3_{\xi\xi\xi}\dot\rho \end{pmatrix}.$$

By Fourier transform, we find that $\lambda\in\mathbb{C}$ belongs to the spectrum of L^σ_∞ if and only if there exists $\zeta\in\mathbb{R}$ such that

$$(\lambda \,+\, i\,(u_\infty-\sigma)\zeta)^2 \,+\, \rho_\infty\left(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) \,+\, K(\rho_\infty)\,\zeta^2\right)\zeta^2 \;=\; 0\,. \tag{2.15}$$

Since by assumption $K(\rho_\infty)>0$, and as we have seen above, $\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)>0$ (a necessary condition for the homoclinic wave to exist), (2.15) has no solution $\zeta\in\mathbb{R}$ for $\lambda\notin i\mathbb{R}$. By standard (Coppel-Palmer [10, 16], or Henry [13]) arguments, this implies that the essential spectrum of the variable-coefficients operator L^σ is contained in $i\mathbb{R}$ (and in fact equal to $i\mathbb{R}$ because all elements of $i\mathbb{R}$ are 'approximate eigenvalues' of L^σ).

• **Construction of the Evans function.** In order to construct an Evans function [1, 17], we first rewrite the eigenvalue equations $(L^\sigma-\lambda)\cdot\dot{\mathbf{U}} \;=\; 0$ as a first order system of ODEs, where ξ is viewed as a 'time'-variable. By definition,

$$L^\sigma\cdot\dot{\mathbf{U}} \;=\; \begin{pmatrix} -\,\partial_\xi\Big((\underline{u}^\sigma-\sigma)\,\dot\rho \,+\, \underline{\rho}^\sigma\,\dot u\Big) \\ \partial_\xi\Big(-(\underline{u}^\sigma-\sigma)\,\dot u \,-\, \underline{\alpha}^\sigma\,\dot\rho \,+\, \underline{K}^\sigma\,\partial^2_{\xi\xi}\dot\rho \,+\, \partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big) \end{pmatrix},$$

where $\underline{K}^\sigma \;:=\; K(\underline{\rho}^\sigma)$ and

$$\underline{\alpha}^\sigma \;:=\; \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\underline{\rho}^\sigma) \;-\; \frac{\mathrm{d}K}{\mathrm{d}\rho}(\underline{\rho}^\sigma)\,\partial^2_{\xi\xi}\underline{\rho}^\sigma \;-\; \frac{1}{2}\,\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2}(\underline{\rho}^\sigma)\,(\partial_\xi\underline{\rho}^\sigma)^2\,.$$

So $(L^\sigma-\lambda)\cdot\dot{\mathbf{U}} \;=\; 0$ is equivalent to

$$(B^\sigma\,\Phi)' \;=\; A(\lambda)\,\Phi\,, \tag{2.16}$$

where the prime ($'$) stands for $\mathrm{d}/\mathrm{d}\xi$, and

$$\Phi := \begin{pmatrix} \dot\rho \\ \dot\rho' \\ \dot\rho'' \\ \dot u \end{pmatrix},\quad B^\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\underline{\alpha}^\sigma & (\underline{K}^\sigma)' & \underline{K}^\sigma & -(\underline{u}^\sigma-\sigma) \\ (\underline{u}^\sigma-\sigma) & 0 & 0 & \underline{\rho}^\sigma \end{pmatrix},\quad A(\lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \\ -\lambda & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the asymptotic system $(B^\sigma_\infty\,\Phi)' \;=\; A(\lambda)\,\Phi$, with

$$B^\sigma_\infty := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c_\infty^2/\rho_\infty & 0 & K_\infty & -(u_\infty-\sigma) \\ (u_\infty-\sigma) & 0 & 0 & \rho_\infty \end{pmatrix},\qquad K_\infty \;:=\; K(\rho_\infty)\,,$$

are the roots ω of the dispersion relation

$$(\lambda \,+\, (u_\infty-\sigma)\omega)^2 \;-\; \left(c_\infty^2 \;-\; \rho_\infty\,K_\infty\,\omega^2\right)\omega^2 \;=\; 0 \tag{2.17}$$

(Alternatively, (2.17) can be derived from (2.15) by substituting ω for $i\zeta$.) We easily see that, for $\mathrm{Re}\lambda>0$, (2.17) has no purely imaginary root ω, and by studying the case $\lambda\in\mathbb{R}$, $\lambda\gg1$, we find that (2.17) has exactly two roots of negative real parts, say $\omega_1(\lambda)$ and $\omega_2(\lambda)$ (either both real or complex conjugate), and two roots of positive real parts, say $\omega_3(\lambda)$ and $\omega_4(\lambda)$ (either both real or complex conjugate). When λ goes to zero, the four roots are real, and two of them go to zero. We choose their numbering so that

$$\omega_2 \sim \frac{-\lambda}{c_\infty + u_\infty - \sigma}\,, \quad \omega_3 \sim \frac{\lambda}{c_\infty - u_\infty + \sigma}\,,$$

$$\omega_1 \to -\sqrt{(c_\infty^2 - (u_\infty - \sigma)^2)/(\rho_\infty K_\infty)}\,, \quad \omega_4 \to +\sqrt{(c_\infty^2 - (u_\infty - \sigma)^2)/(\rho_\infty K_\infty)}$$

when λ goes to 0. In addition, at points λ where ω_1 and ω_2 are distinct, respectively where ω_3 and ω_4 are distinct, which is the case for large real λ and for λ close to zero, the corresponding eigenvectors, $\mathbf{W}_1^\sigma(\lambda)$, $\mathbf{W}_2^\sigma(\lambda)$, and respectively $\mathbf{W}_3^\sigma(\lambda)$, $\mathbf{W}_4^\sigma(\lambda)$, span the stable, and respectively the unstable, subspace (in $\mathbb{C}^4$) of the matrix $(B_\infty^\sigma)^{-1}A(\lambda)$. They can be chosen of the form

$$\mathbf{W}_j^\sigma(\lambda) := \begin{pmatrix} \rho_\infty \\ \rho_\infty\,\omega_j^\sigma(\lambda) \\ \rho_\infty\,\omega_j^\sigma(\lambda)^2 \\ -\frac{\lambda}{\omega_j^\sigma(\lambda)} - (u_\infty - \sigma) \end{pmatrix}. \tag{2.18}$$

Then their limits at $\lambda = 0$ are easily found to be

$$\mathbf{W}_{1,4}^\sigma(0) = \begin{pmatrix} \rho_\infty \\ \rho_\infty\,\omega_{1,4}^\sigma(0) \\ \rho_\infty\,\omega_{1,4}^\sigma(0)^2 \\ -(u_\infty - \sigma) \end{pmatrix}, \quad \mathbf{W}_2^\sigma(0) = \begin{pmatrix} \rho_\infty \\ 0 \\ 0 \\ c_\infty \end{pmatrix}, \quad \mathbf{W}_3^\sigma(0) = \begin{pmatrix} \rho_\infty \\ 0 \\ 0 \\ -c_\infty \end{pmatrix}. \tag{2.19}$$

We can construct a so-called Evans function D^σ, which is analytic and real valued for $\lambda \in [0, +\infty)$, such that

$$D^\sigma(\lambda) = 0\,,\ \lambda > 0 \quad \Longleftrightarrow \quad \mathrm{Ker}\,(L^\sigma - \lambda) \neq \{0\}\,.$$

(See [1, 17] for $\lambda > 0$, and [11, 15] for the extension to $\lambda = 0$.) More precisely, D^σ can be taken of the form

$$D^\sigma(\lambda) = \det(\widetilde{\Phi}_1^\sigma(\lambda), \widetilde{\Phi}_2^\sigma(\lambda), \widetilde{\Phi}_3^\sigma(\lambda), \widetilde{\Phi}_4^\sigma(\lambda))|_{\xi=0}\,,$$

where $(\widetilde{\Phi}_1^\sigma(\lambda), \widetilde{\Phi}_2^\sigma(\lambda))$ (respectively $(\widetilde{\Phi}_3^\sigma(\lambda), \widetilde{\Phi}_3^\sigma(\lambda))$), span the real stable (respectively unstable) manifold of (2.16). These real-valued $\widetilde{\Phi}_j^\sigma$ can be constructed in a simple way from the complex-valued solutions Φ_j^σ of (2.16) characterized, at nonglancing points, by

$$\Phi_{1,2}^\sigma(\lambda) \overset{\xi\to+\infty}{\sim} \mathrm{e}^{\omega_{1,2}^\sigma(\lambda)\xi}\,\mathbf{W}_{1,2}^\sigma(\lambda)\,, \quad \Phi_{3,4}^\sigma(\lambda) \overset{\xi\to-\infty}{\sim} \mathrm{e}^{\omega_{3,4}^\sigma(\lambda)\xi}\,\mathbf{W}_{3,4}^\sigma(\lambda)\,. \tag{2.20}$$

It suffices to define

$$\widetilde{\Phi}_1^\sigma := \Phi_1^\sigma + \Phi_2^\sigma\,, \quad \widetilde{\Phi}_2^\sigma := \frac{\Phi_1^\sigma - \Phi_2^\sigma}{\omega_1 - \omega_2}\,,$$

$$\widetilde{\Phi}_3^\sigma := \Phi_3^\sigma + \Phi_4^\sigma\,, \quad \widetilde{\Phi}_4^\sigma := \frac{\Phi_3^\sigma - \Phi_4^\sigma}{\omega_3 - \omega_4}\,.$$

These $\widetilde{\Phi}_j^\sigma$s, as the Φ_j^σs, depend analytically on λ away from glancing points. Furthermore, they are obviously real-valued when the Φ_j^σs are so. Otherwise, when (ω_1, ω_2) is a conjugate pair, so is $(\Phi_1^\sigma, \Phi_2^\sigma)$ and therefore the $\widetilde{\Phi}_{1,2}^\sigma$ are still real-valued. Of course the same observation holds true with the indices $(3, 4)$ instead of $(1, 2)$. Note also that the $\widetilde{\Phi}_j^\sigma$s do not depend on the numbering of stable and unstable modes. As usual, it is trickier to define the Evans function at glancing points, that is, where either ω_1 and

ω_2, or ω_3 and ω_4, collide (which does happen, as a closer examination of the algebraic equation (2.17) shows). Indeed, even though the eigenvectors $\mathbf{W}^\sigma_j$ are such that

$$\widetilde{\mathbf{W}}^\sigma_2 := \frac{\mathbf{W}^\sigma_1 - \mathbf{W}^\sigma_2}{\omega_1 - \omega_2} \quad \text{and} \quad \widetilde{\mathbf{W}}^\sigma_4 := \frac{\mathbf{W}^\sigma_3 - \mathbf{W}^\sigma_4}{\omega_3 - \omega_4}$$

do have limits at glancing points that are independent of $\mathbf{W}^\sigma_2$ and $\mathbf{W}^\sigma_4$ respectively (as is easily found from (2.18)), which means that $(B^\sigma_\infty)^{-1}A(\lambda)$ has a 2×2 Jordan block at those points), the behavior of the individual $\widetilde{\Phi}^\sigma_{2,4}$ is unclear. However, working with wedge products [1] we can make sure that the Evans function crosses glancing points in a continuous (and even analytic) manner.

• **Low frequency expansion of the Evans function.** Observing that by definition

$$D^\sigma(\lambda) \;=\; \frac{\det(\Phi^\sigma_1(\lambda), \Phi^\sigma_2(\lambda), \Phi^\sigma_3(\lambda), \Phi^\sigma_4(\lambda))|_{\xi=0}}{(\omega_2(\lambda) - \omega_1(\lambda))(\omega_4(\lambda) - \omega_3(\lambda))}\,,$$

where the denominator in the neighborhood of $\lambda = 0$ is

$$(\omega_2(\lambda) - \omega_1(\lambda))(\omega_4(\lambda) - \omega_3(\lambda)) \sim \frac{c^2_\infty - (u_\infty - \sigma)^2}{\rho_\infty K_\infty} \;>\; 0\,,$$

we see that $D^\sigma(\lambda)$ has the same sign as

$$\Delta^\sigma(\lambda) \;:=\; \det(\Phi^\sigma_1(\lambda), \Phi^\sigma_2(\lambda), \Phi^\sigma_3(\lambda), \Phi^\sigma_4(\lambda))|_{\xi=0} \tag{2.21}$$

for λ close to 0.

Since $L^\sigma \cdot (\underline{\mathbf{U}}^\sigma)' \;=\; 0$ and $(\underline{\mathbf{U}}^\sigma)'$ goes exponentially fast to zero at $\pm\infty$, the one-dimensional stable/unstable manifold of (2.16) with $\lambda = 0$ is spanned by $(\underline{\mathbf{U}}^\sigma)'$. This means that both $\Phi^\sigma_1(0)$ and $\Phi^\sigma_4(0)$ must be proportional to $(\underline{\mathbf{U}}^\sigma)'$. Now we have to be careful to comply with (2.18) and (2.20), which imply in particular that the first component of $\Phi^\sigma_1(0)$, respectively $\Phi^\sigma_4(0)$, must be positive when ξ goes to $+\infty$, respectively $-\infty$. Since $(\underline{\rho}^\sigma)'$ has different signs at $+\infty$ and $-\infty$, this means there exists a nonzero real number r such that

$$\Phi^\sigma_1(0) \;=\; -\,r \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{\rho}^\sigma)'' \\ (\underline{\rho}^\sigma)''' \\ (\underline{u}^\sigma)' \end{pmatrix}, \quad \Phi^\sigma_4(0) \;=\; r \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{\rho}^\sigma)'' \\ (\underline{\rho}^\sigma)''' \\ (\underline{u}^\sigma)' \end{pmatrix}. \tag{2.22}$$

The actual value of r can be deduced from the phase portrait of the profile equation (which is symmetric with respect to the horizontal axis), its sign depending on the type of soliton considered. It is of no importance though. We only need to know that the sign of $D^\sigma(\lambda)$ (for small λ) is opposite to the sign of

$$\breve{\Delta}^\sigma(\lambda) \;:=\; \det(\breve{\Phi}^\sigma_1(\lambda), \Phi^\sigma_2(\lambda), \Phi^\sigma_3(\lambda), \breve{\Phi}^\sigma_4(\lambda))|_{\xi=0} \quad \breve{\Phi}_1 := -(1/r)\,\Phi_1\,, \quad \breve{\Phi}_4 := (1/r)\,\Phi_4.$$

Taking (2.22) into account in (2.21) we readily find that $\breve{\Delta}^\sigma(0) = 0$. Furthermore, $(\breve{\Delta}^\sigma)'(0) = 0$. This can be seen as follows. Denoting by $\phi^\sigma_j(\lambda)$ and $\mu^\sigma_j(\lambda)$ the first and fourth components of $\Phi^\sigma_j(\lambda)$ (or $\breve{\Phi}^\sigma_j(\lambda)$ for $j = 1$ or 4) respectively, we find by differentiation of $(B^\sigma\,\Phi^\sigma_j(\lambda))' \;=\; A(\lambda)\,\Phi^\sigma_j(\lambda)$ with respect to λ that, thanks to (2.22) and (2.9),

$$L^\sigma \cdot \begin{pmatrix} \partial_\lambda \phi^\sigma_{1,4}(0) \\ \partial_\lambda \mu^\sigma_{1,4}(0) \end{pmatrix} \;=\; \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix} \;=\; -\,L^\sigma \cdot \begin{pmatrix} \partial_\sigma \underline{\rho}^\sigma \\ \partial_\sigma \underline{u}^\sigma \end{pmatrix},$$

which implies

$$\begin{pmatrix} \partial_\lambda \phi^\sigma_{1,4}(0) + \partial_\sigma \underline{\rho}^\sigma \\ \partial_\lambda \mu^\sigma_{1,4}(0) + \partial_\sigma \underline{u}^\sigma \end{pmatrix} \parallel \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix}, \text{ a generator of the one-dimensional kernel of } L^\sigma\,.$$

Therefore, using (2.22) again and up to adding a constant times $\lambda\Phi^\sigma_{1,4}(\lambda)$ to $\Phi^\sigma_{1,4}(\lambda)$, we may assume without loss of generality that

$$\partial_\lambda\breve{\Phi}^\sigma_1(0) \,=\, \partial_\lambda\breve{\Phi}^\sigma_4(0) \,=\, -\begin{pmatrix} \partial_\sigma(\underline{\rho}^\sigma) \\ \partial_\sigma(\underline{\rho}^\sigma)' \\ \partial_\sigma(\underline{\rho}^\sigma)'' \\ \partial_\sigma(\underline{u}^\sigma) \end{pmatrix}. \tag{2.23}$$

Together with (2.22), this obviously implies that $(\breve{\Delta}^\sigma)'(0) \,=\, 0$. Differentiating once more, we find that

$$(\breve{\Delta}^\sigma)''(0) \,=\, \det(\breve{\Phi}^\sigma_1(0), \Phi^\sigma_2(0), \Phi^\sigma_3(0), \partial^2_{\lambda\lambda}(\breve{\Phi}^\sigma_4 - \breve{\Phi}^\sigma_1)(0))|_{\xi=0}\,.$$

To evaluate this determinant, we first observe that $\det B^\sigma|_{\xi=0} \,=\, \underline{\rho}^\sigma(0)\,\underline{K}^\sigma(0) \,\neq\, 0$, so that

$$\det(\breve{\Phi}^\sigma_1(0), \Phi^\sigma_2(0), \Phi^\sigma_3(0), \partial^2_{\lambda\lambda}(\breve{\Phi}^\sigma_4 - \breve{\Phi}^\sigma_1)(0))|_{\xi=0}$$

$$=\, \frac{1}{\underline{\rho}^\sigma(0)\,\underline{K}^\sigma(0)}\ \det(B^\sigma\breve{\Phi}^\sigma_1(0), B^\sigma\Phi^\sigma_2(0), B^\sigma\Phi^\sigma_3(0), \partial^2_{\lambda\lambda}B^\sigma(\breve{\Phi}^\sigma_4 - \breve{\Phi}^\sigma_1)(0))|_{\xi=0}\,.$$

For simplicity, in what follows, we just denote by Φ_j the function $\Phi^\sigma_j(0)$, and by ϕ_j and μ_j its first and last components, and $\Theta_j \,=\, \partial^2_{\lambda\lambda}\breve{\Phi}^\sigma_j(0)$, with θ_j and χ_j its first and last components. By construction of Φ_j, since the last two rows of $A(0)$ are zero, we have

$$B^\sigma\,\Phi_j \,=\, \begin{pmatrix} \phi_j \\ \phi_j' \\ R_j \end{pmatrix},$$

where R_j is a *constant vector* in $\mathbb{R}^2$. More specifically, R_1 is the null vector, while

$$\lim_{\xi\to+\infty}\phi_2(\xi) \,=\, \rho_\infty\,,\ \lim_{\xi\to+\infty}\mu_2(\xi) \,=\, c_\infty\,,\quad \lim_{\xi\to-\infty}\phi_3(\xi) \,=\, \rho_\infty\,,\ \lim_{\xi\to-\infty}\mu_3(\xi) \,=\, -\,c_\infty$$

(which come from (2.19) and (2.20)), imply that

$$R_2 \,=\, \begin{pmatrix} -c_\infty(u_\infty - \sigma + c_\infty) \\ \rho_\infty(u_\infty - \sigma + c_\infty) \end{pmatrix},\quad R_3 \,=\, \begin{pmatrix} c_\infty(u_\infty - \sigma - c_\infty) \\ \rho_\infty(u_\infty - \sigma - c_\infty) \end{pmatrix}.$$

Furthermore, we claim that

$$B^\sigma\,\Theta_{1,4} \,=\, \begin{pmatrix} \theta_{1,4} \\ \theta_{1,4}' \\ S_{1,4} \end{pmatrix},$$

with $S_{1,4} : \xi \to S_{1,4}(\xi) \in \mathbb{R}^2$ such that

$$S_4 - S_1 \,=\, 2\int_{-\infty}^{+\infty}\begin{pmatrix} -\partial_\sigma\underline{u}^\sigma \\ \partial_\sigma\underline{\rho}^\sigma \end{pmatrix}\mathrm{d}\xi\,. \tag{2.24}$$

Indeed, differentiating twice $(B^\sigma\,\Phi^\sigma_j(\lambda))' \,=\, A(\lambda)\,\Phi^\sigma_j(\lambda)$ with respect to λ at $\lambda = 0$, and using (2.23), we find that

$$L^\sigma\cdot\begin{pmatrix} \theta_{1,4} \\ \chi_{1,4} \end{pmatrix} \,=\, -\,2\begin{pmatrix} \partial_\sigma\underline{\rho}^\sigma \\ \partial_\sigma\underline{u}^\sigma \end{pmatrix},$$

hence

$$\begin{cases} (\underline{u}^\sigma - \sigma)\,\theta_1 \,+\, \underline{\rho}^\sigma\,\chi_1 \,=\, -\,2\int_\xi^{+\infty}\partial_\sigma\underline{\rho}^\sigma\,, \\ \underline{K}^\sigma\,\theta_1'' \,+\, (\underline{K}^\sigma)'\,\theta_1' \,-\, \underline{\alpha}^\sigma\,\theta_1 \,-\, (\underline{u}^\sigma - \sigma)\,\chi_1 \,=\, 2\int_\xi^{+\infty}\partial_\sigma\underline{u}^\sigma\,, \end{cases}$$

$$
\begin{cases} (\underline{u}^\sigma - \sigma)\,\theta_4 \,+\, \underline{\rho}^\sigma\,\chi_4 \,=\, 2\,\int_{-\infty}^{\xi} \partial_\sigma \underline{\rho}^\sigma\ , \\ \underline{K}^\sigma\,\theta_4'' \,+\, (\underline{K}^\sigma)'\,\theta_4' \,-\, \underline{\alpha}^\sigma\,\theta_4 \,-\, (\underline{u}^\sigma - \sigma)\,\chi_4 \,=\, -\,2\,\int_{-\infty}^{\xi} \partial_\sigma \underline{u}^\sigma\ , \end{cases}
$$

which imply (2.24) by definition of S_1 and S_4. To complete the computation of $(\check{\Delta}^\sigma)''(0)$, we observe that

$$
\det(R_2, R_3) \,=\, 2\,\rho_\infty\,c_\infty\,(c_\infty^2 \,-\, (u_\infty - \sigma)^2) \,>\, 0
$$

by (2.12), and we introduce (the unique) real numbers d_2 and d_3 such that

$$
S_4 - S_1 \,=\, d_2\,R_2 \,-\, d_3\,R_3\,.
$$

Therefore,

$$
(\check{\Delta}^\sigma)''(0) \,=\, \frac{1}{\underline{\rho}^\sigma(0)\,\underline{K}^\sigma(0)} \left| \begin{array}{cccc} (\underline{\rho}^\sigma)' & \phi_2 & \phi_3 & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ (\underline{\rho}^\sigma)'' & \phi_2' & \phi_3' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \\ 0_2 & R_2 & R_3 & 0_2 \end{array} \right|_{|\xi=0} \,=\, \frac{\det(R_2,R_3)}{\underline{\rho}^\sigma(0)\,\underline{K}^\sigma(0)} \left| \begin{array}{cc} (\underline{\rho}^\sigma)' & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \end{array} \right|_{|\xi=0}
$$

with

$$
\widetilde{\theta}_4 \,:=\, \theta_4 \,+\, d_3\,\phi_3\,, \quad \widetilde{\theta}_1 \,:=\, \theta_1 \,+\, d_2\,\phi_2\,.
$$

It thus only remains to compute $\delta_{|\xi=0}$, with

$$
\delta \,:=\, \left| \begin{array}{cc} (\underline{\rho}^\sigma)' & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \end{array} \right|,
$$

knowing that $(\underline{\rho}^\sigma)'$ and $\widetilde{\theta}_{1,4}$ all satisfy an ODE of the form

$$
\underline{K}^\sigma\,y'' \,+\, (\underline{K}^\sigma)'\,y' \,-\, \underline{\alpha}^\sigma\,y \,+\, \frac{1}{\underline{\rho}^\sigma}(\underline{u}^\sigma - \sigma)^2\,y \,=\, s[y]\,,
$$

and more precisely,

$$
s[(\underline{\rho}^\sigma)'] \,=\, 0\,, \quad s[\widetilde{\theta}_4] \,=\, (1, (\underline{u}^\sigma - \sigma)/\underline{\rho}^\sigma)\,(S_4 \,+\, d_3\,R_3) \,=\, (1, (\underline{u}^\sigma - \sigma)/\underline{\rho}^\sigma)\,(S_1 \,+\, d_2\,R_2) \,=\, s[\widetilde{\theta}_1]\,.
$$

The rest of the computation is based on the Melnikov technique. Decomposing δ as

$$
\delta \,=\, \delta_4 - \delta_1\,, \quad \delta_{1,4} \,:=\, \left| \begin{array}{cc} (\underline{\rho}^\sigma)' & \widetilde{\theta}_{1,4} \\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}_{1,4}' \end{array} \right|, \quad \text{with } \delta_4(-\infty) \,=\, 0\,,\ \delta_1(+\infty) \,=\, 0\,,
$$

and integrating the ODEs

$$
\frac{\mathrm{d}\delta_{1,4}}{\mathrm{d}\xi} \,=\, -\,\frac{(\underline{K}^\sigma)'}{\underline{K}^\sigma}\,\delta_{1,4} \,+\, \frac{(\underline{\rho}^\sigma)'}{\underline{K}^\sigma}\,s[\widetilde{\theta}_{1,4}]
$$

on $(0,+\infty)$ and $(-\infty,0)$ respectively, we find that

$$
\delta_{|\xi=0} \,=\, \frac{1}{\underline{K}^\sigma(0)} \int_{-\infty}^{+\infty} s[\widetilde{\theta}_{1,4}]\,(\underline{\rho}^\sigma)'\,.
$$

Now, thanks to the identity

$$
(\underline{u}^\sigma - \sigma)\,(\underline{\rho}^\sigma)' \,=\, -\underline{\rho}^\sigma\,(\underline{u}^\sigma)'\,,
$$

we have

$$
\int_{-\infty}^{+\infty} s[\widetilde{\theta}_4]\,(\underline{\rho}^\sigma)' \,=\, \int_{-\infty}^{+\infty} ((\underline{\rho}^\sigma)', -(\underline{u}^\sigma)')\,(S_4 \,+ d_3\,R_3)\,.
$$

Clearly (since $\underline{\rho}^\sigma$, $\underline{u}^\sigma$ have the same limits at $+\infty$ and $-\infty$) the constant vector R_3 does not contribute to this integral, and recalling that

$$S_4(\xi) \,=\, 2\int_{-\infty}^{\xi} \left(\begin{array}{c} -\partial_\sigma \underline{u}^\sigma \\ \partial_\sigma \underline{\rho}^\sigma \end{array} \right),$$

after integration by parts we finally arrive at

$$\delta_{|\xi=0} \,=\, \frac{2}{\underline{K}^\sigma(0)} \int_{-\infty}^{+\infty} \Big((\underline{\rho}^\sigma - \rho_\infty)\, \partial_\sigma \underline{u}^\sigma \,+\, (\underline{u}^\sigma - u_\infty)\, \partial_\sigma \underline{\rho}^\sigma \Big)\,.$$

This yields the formula

$$(\breve{\Delta}^\sigma)''(0) \,=\, \frac{4\,\rho_\infty\, c_\infty\, (c_\infty^2 \,-\, (u_\infty - \sigma)^2)}{\underline{\rho}^\sigma(0)(\underline{K}^\sigma(0))^2} \int_{-\infty}^{+\infty} \Big((\underline{\rho}^\sigma - \rho_\infty)\, \partial_\sigma \underline{u}^\sigma \,+\, (\underline{u}^\sigma - u_\infty)\, \partial_\sigma \underline{\rho}^\sigma \Big)\,,$$

hence

$$(\Delta^\sigma)''(0) \,=\, -\,\frac{4\,r^2\,\rho_\infty\, c_\infty\, (c_\infty^2 \,-\, (u_\infty - \sigma)^2)}{\underline{\rho}^\sigma(0)(\underline{K}^\sigma(0))^2} \int_{-\infty}^{+\infty} \Big((\underline{\rho}^\sigma - \rho_\infty)\, \partial_\sigma \underline{u}^\sigma \,+\, (\underline{u}^\sigma - u_\infty)\, \partial_\sigma \underline{\rho}^\sigma \Big)\,.$$

• **High frequency behavior of the Evans function.** This part of the analysis could be omitted – and is indeed omitted in [21] – in view of the sufficient stability condition provided by the Grillakis-Shatah-Strauss method. It is of interest though, for the method – which can be useful in other frameworks –, and as a way to double-check that the stability condition is indeed

$$\int \Big((\underline{\rho}^\sigma - \rho_\infty)\, \partial_\sigma \underline{u}^\sigma \,+\, (\underline{u}^\sigma - u_\infty)\, \partial_\sigma \underline{\rho}^\sigma \Big)\, \mathrm{d}\xi \,<0\,. \tag{2.25}$$

By means of an energy estimate based on a 'symmetrized' reformulation of the linearized system (see [7], Proposition 3.4), we can find $\lambda_0 > 0$ such that L^σ has no eigenvalue $\lambda > \lambda_0$. We may then argue by homotopy. For $\theta \in [0,1]$, consider the operator the operator L_θ^σ defined by

$$L_\theta^\sigma \cdot \dot{\mathbf{U}} \,=\, \left(\begin{array}{l} -\,\partial_\xi \Big(u_\theta^\sigma\, \dot\rho \,+\, \rho_\theta^\sigma\, \dot u \Big) \\ \partial_\xi \Big(-u_\theta^\sigma\, \dot u \,-\, \alpha_\theta^\sigma\, \dot\rho \,+\, K_\theta^\sigma\, \partial_{\xi\xi}^2 \dot\rho \,+\, \partial_\xi K_\theta^\sigma\, \partial_\xi \dot\rho \Big) \end{array} \right),$$

where

$$u_\theta^\sigma := \theta\,(\underline{u}^\sigma - \sigma)\,,\quad \rho_\theta^\sigma := \rho_\infty + \theta(\underline{\rho}^\sigma - \rho_\infty)\,,\quad K_\theta^\sigma \,:=\, K(\rho_\theta^\sigma)\,,$$

$$\alpha_\theta^\sigma \,:=\, \theta\, \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\theta^\sigma) \,-\, \frac{\mathrm{d}K}{\mathrm{d}\rho}(\rho_\theta^\sigma)\, \partial_{\xi\xi}^2 \rho_\theta^\sigma \,-\, \frac{1}{2}\, \frac{\mathrm{d}^2 K}{\mathrm{d}\rho^2}(\rho_\theta^\sigma)\, (\partial_\xi \rho_\theta^\sigma)^2\,.$$

At $\theta = 1$ we recover L^σ and at $\theta = 0$ we get the constant-coefficients operator

$$L_0 \cdot \dot{\mathbf{U}} \,:=\, \left(\begin{array}{c} -\,\rho_\infty\, \partial_\xi \dot u \\ K_\infty\, \partial_{\xi\xi\xi}^3 \dot\rho \end{array} \right).$$

The spectrum of L_0 is found to be exactly $i\mathbb{R}$ by Fourier transform. Furthermore, the aforementioned energy estimate can be adapted to deal with L_θ^σ and show that for all $\theta \in [0,1]$, L_θ^σ has no eigenvalue of real part greater than some threshold $\lambda_* \geq \lambda_0$. Let us describe how to obtain this estimate, which is not straightforward. Assume that $\dot{\mathbf{U}} = (\dot\rho, \dot u)^{\mathrm{t}}$ is an eigenvector associated with a nonzero eigenvalue λ of L_θ^σ (viewed as an unbounded operator on $H^1 \times L^2$ with domain $H^3 \times H^2$). We look for a λ_* independent of $\dot{\mathbf{U}}$ and θ such that

$$(\mathrm{Re}\lambda - \lambda_*)\, \|\dot{\mathbf{U}}\|_{H^1 \times L^2}^2 \,\leq 0\,.$$

Since the principal part of L_θ^σ is not dissipative, the elimination of higher order derivatives is not straightforward. It requires a 'symmetrized' reformulation of the eigenvalue equation $(\lambda - L_\theta^\sigma)\dot{\mathbf{U}} = \mathbf{0}$. As observed in earlier work [4, 5], a suitable reformulation makes use of the change of variables $\rho \mapsto \zeta := R(\rho)$, where R is a primitive of $\rho \mapsto \sqrt{K(\rho)/\rho}$, which urges us to consider $\dot\zeta := R'(\rho_\theta^\sigma)\dot\rho$, and derive an estimate for $\|\dot\zeta\|_{L^2} + \|\sqrt{\rho_\theta^\sigma}\,\dot u\|_{L^2} + \|\sqrt{\rho_\theta^\sigma}\,\dot w\|_{L^2}$ with $\dot w := \partial_\xi\dot\zeta$ instead of the standard norm $\|\dot{\mathbf{U}}\|_{H^1\times L^2}$. We first compute the system satisfied by $(\dot\zeta, \dot u, \dot w)$ if $(\lambda - L_\theta^\sigma)\dot{\mathbf{U}} = \mathbf{0}$. Introducing the functions a and h defined by $a(\zeta) := \sqrt{R^{-1}(\zeta)\,K(R^{-1}(\zeta))}$ and $h(\zeta) := \frac{\mathrm{d}}{\mathrm{d}\zeta}\,g_0(R^{-1}(\zeta))$, we can write this system as

$$\lambda\,\dot\zeta \,+\, u_\theta^\sigma\,\dot w \,+\, \dot u\,w_\theta^\sigma \,+\, a_\theta^\sigma\,\partial_\xi\dot u \,+\, (a_\theta^\sigma)'\,\dot\zeta\,\partial_\xi u_\theta^\sigma = 0\,, \tag{2.26}$$

$$\lambda\,\dot u \,+\, \partial_\xi\left(u_\theta^\sigma\,\dot u \,-\, w_\theta^\sigma\,\dot w \,-\, a_\theta^\sigma\,\partial_\xi\dot w \,-\, (a_\theta^\sigma)'\,\dot\zeta\,\partial_\xi w_\theta^\sigma\right) \,+\, h_\theta^\sigma\,\dot w \,+\, (h_\theta^\sigma)'\,\dot\zeta\,w_\theta^\sigma \,=\, 0\,, \tag{2.27}$$

$$\lambda\,\dot w \,+\, \partial_\xi(u_\theta^\sigma\,\dot w \,+\, \dot u\,w_\theta^\sigma) \,+\, \partial_\xi\left(a_\theta^\sigma\,\partial_\xi\dot u \,+\, (a_\theta^\sigma)'\,\dot\zeta\,\partial_\xi u_\theta^\sigma)\right) \,=0\,, \tag{2.28}$$

where $\zeta_\theta^\sigma := R(\rho_\theta^\sigma)$, $w_\theta^\sigma := R'(\rho_\theta^\sigma)\,\partial_\xi\rho_\theta^\sigma$, $a_\theta^\sigma := a(\zeta_\theta^\sigma)$, $(a_\theta^\sigma)' := \frac{\mathrm{d}a}{\mathrm{d}\zeta}(\zeta_\theta^\sigma)$, $h_\theta^\sigma := h(\zeta_\theta^\sigma)$, $(h_\theta^\sigma)' := \frac{\mathrm{d}h}{\mathrm{d}\zeta}(\zeta_\theta^\sigma)$. Interestingly, (2.27) and (2.28) can be written as a single equation for the complex-valued function $\dot z := \dot u + i\dot w$,

$$\lambda\,\dot z \,+\, \partial_\xi\left(z_\theta^\sigma\,\dot z \,+\, i\,a_\theta^\sigma\,\partial_\xi\dot z \,+\, i\,(a_\theta^\sigma)'\,\dot\zeta\,\partial_\xi z_\theta^\sigma\right) \,+\, h_\theta^\sigma\,\dot w \,+\, (h_\theta^\sigma)'\,\dot\zeta\,w_\theta^\sigma \,=\, 0\,, \tag{2.29}$$

where $z_\theta^\sigma := u_\theta^\sigma + iw_\theta^\sigma$. Taking the real part of the inner product of (2.29) with $\rho_\theta^\sigma\dot z$, integrating by part, and using that $a_\theta^\sigma\,\partial_\xi\rho_\theta^\sigma \,=\, \rho_\theta^\sigma\,w_\theta^\sigma$, we get

$$\mathrm{Re}\lambda\,\|\sqrt{\rho_\theta^\sigma}\,\dot z\|_{L^2}^2 \,+\, \mathrm{Re}\langle(\partial_\xi z_\theta^\sigma)\,\dot z, \rho_\theta^\sigma\dot z\rangle \,+\, \langle\partial_\xi(\rho_\theta^\sigma\,u_\theta^\sigma)\,\dot z, \dot z\rangle +$$
$$\mathrm{Re}\langle(h_\theta^\sigma + i(a_\theta^\sigma)'\,\partial_\xi z_\theta^\sigma)\,\dot w, \rho_\theta^\sigma\dot z\rangle \,+\, \mathrm{Re}\langle((h_\theta^\sigma)' + i\partial_\xi((a_\theta^\sigma)'\,\partial_\xi z_\theta^\sigma))\,\dot\zeta, \rho_\theta^\sigma\dot z\rangle \,=\, 0\,.$$

(Without the weight ρ_θ^σ there would have remained a term with the first-order derivative $\partial_\xi\dot z$: this is reminiscent of the symmetrization issue for Euler equations.) On the other hand, taking the real part of the inner product of (2.26) with $\dot\zeta$ we obtain

$$\mathrm{Re}\lambda\,\|\dot\zeta\|_{L^2}^2 \,+\, \mathrm{Re}\langle u_\theta^\sigma\,\dot w, \dot\zeta\rangle \,+\, \mathrm{Re}\langle(w_\theta^\sigma - (a_\theta^\sigma)'\,\partial_\xi\zeta_\theta^\sigma)\,\dot u, \dot\zeta\rangle \,-\, \mathrm{Re}\langle a_\theta^\sigma\,\dot u, \dot w\rangle \,+\, \mathrm{Re}\langle(a_\theta^\sigma)'\,\dot\zeta\,\partial_\xi u_\theta^\sigma, \dot\zeta\rangle = 0\,.$$

Summing these two identities we find indeed by Cauchy-Schwarz a λ_* (depending only on the $W^{2,\infty}$ norm of $(\zeta_\theta^\sigma, z_\theta^\sigma)$, which is uniformly bounded for $\theta \in [0,1]$) such that

$$(\mathrm{Re}\lambda \,-\, \lambda_*)\,(\,\|\dot\zeta\|_{L^2}^2 \,+\, \|\sqrt{\rho_\theta^\sigma}\,\dot z\|_{L^2}^2) \,\leq\, 0\,,$$

which obviously implies, if $\dot{\mathbf{U}}$, and thus $(\dot\zeta, \dot z)$ is nonzero, that $\mathrm{Re}\lambda \leq \lambda_*$.

Now, we can construct an Evans function, D_θ^σ say, depending smoothly on θ, and determine the sign of $D^\sigma \,=\, D_1^\sigma$ for $\lambda > \lambda_*$ by computing the sign of D_0^σ, which is constant on $[0, +\infty)$. Denoting by $\omega_j^\sigma(\lambda;\theta)$ the eigenvalues of $(B_{\theta,\infty}^\sigma)^{-1}A(\lambda)$, with

$$B_{\theta,\infty}^\sigma \,:=\, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\theta c_\infty^2/\rho_\infty & 0 & K_\infty & -\theta(u_\infty - \sigma) \\ \theta(u_\infty - \sigma) & 0 & 0 & \rho_\infty \end{pmatrix}\,,$$

and by $\mathbf{W}_j^\sigma(\lambda;\theta)$ the associated eigenvectors,

$$\mathbf{W}_j^\sigma(\lambda;\theta) \,:=\, \begin{pmatrix} \rho_\infty \\ \rho_\infty\,\omega_j^\sigma(\lambda;\theta) \\ \rho_\infty\,\omega_j^\sigma(\lambda;\theta)^2 \\ -\frac{\lambda}{\omega_j^\sigma(\lambda;\theta)} - \theta(u_\infty - \sigma) \end{pmatrix}\,, \tag{2.30}$$

we can find $\Phi_j^\sigma(\lambda;\theta)$, solutions of the first-order ODE equivalent to $(L_\theta^\sigma - \lambda)\dot{\mathbf{U}} = 0$, characterized by their asymptotic behavior as in (2.20). In particular, for $\theta = 0$ they are explicitly given by

$$\Phi_j^\sigma(\lambda;0) \;=\; \mathrm{e}^{\omega_j^\sigma(\lambda;0)\xi}\,\mathbf{W}_j^\sigma(\lambda;0)\,,$$

with the $\omega_j^\sigma(\lambda;0)$ occurring in complex conjugate pairs such that

$$\sum_{j=1}^{4}\omega_j^\sigma(\lambda;0) = 0\,.$$

(Indeed, they are roots of $\lambda^2 + \rho_\infty K_\infty \omega^4 = 0$.) Therefore, we have

$$D_0^\sigma(\lambda) \;=\; \det\left(\mathbf{V}_1+\mathbf{V}_2, \frac{\mathbf{V}_2-\mathbf{V}_1}{\nu_2-\nu_1}, \mathbf{V}_3+\mathbf{V}_4, \frac{\mathbf{V}_4-\mathbf{V}_3}{\nu_4-\nu_3}\right)\,,$$

where ν_j and $\mathbf{V}_j$ are simplifying notations for $\omega_j^\sigma(\lambda;0)$ and $\mathbf{W}_j^\sigma(\lambda;0)$ respectively. The ν_j are of the form $\pm(1\pm i)\upsilon$ with

$$\upsilon := \sqrt{\frac{\lambda}{2\sqrt{\rho_\infty K_\infty}}}\,.$$

Recall that the ordering of ν_1 and ν_2, and of ν_3 and ν_4, does not play any role. To fix the ideas, we can take

$$\nu_1 \;=\; -(1+i)\upsilon\,,\;\nu_2 \;=\; (-1+i)\upsilon\,,\;\nu_3 \;=\; (1-i)\upsilon\,,\;\nu_4 \;=\; (1+i)\upsilon\,.$$

Then

$$D_0^\sigma(\lambda) \;=\; 4\,\rho_\infty^3\,\lambda\,\begin{vmatrix} 1 & 0 & 1 & 0 \\ \mathrm{Re}\nu_1 & 1 & \mathrm{Re}\nu_3 & 1 \\ \mathrm{Re}(\nu_1^2) & \nu_1+\nu_2 & \mathrm{Re}(\nu_3^2) & \nu_3+\nu_4 \\ -\mathrm{Re}(\frac{1}{\nu_1}) & \frac{1}{\nu_1\nu_2} & -\mathrm{Re}(\frac{1}{\nu_3}) & \frac{1}{\nu_3\nu_4}\end{vmatrix} \;=\; 32\,\rho_\infty^3\,\lambda \;>\; 0\,.$$

3 Multi-dimensional stability criterion

The Grillakis-Shatah-Strauss argument invoked for one-dimensional (orbital) stability breaks down in several space dimensions because planar solitary waves do not have an interpretation in terms of critical points. However, the form of the linearized system makes it possible to extend the Evans function calculation of Lemma 1, and eventually show that one-d stable planar solitary waves are unstable with respect to transverse perturbations.

3.1 The linearized operator

By definition, the profile $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ of a planar solitary wave solution of (1.1) propagating in direction $\mathbf{n}$ (a unitary vector in $\mathbb{R}^d$) with speed σ and homoclinic to $(\rho_\infty, \mathbf{u}_\infty)$, must satisfy

(3.31)
$$\left\{\begin{array}{l} \underline{\rho}^\sigma\,(\underline{u}^\sigma \,-\, \sigma) \;\equiv\; \rho_\infty\,(u_\infty \,-\, \sigma)\,, \\[2ex] (\underline{u}^\sigma \,-\, \sigma)\,\partial_\xi\underline{\mathbf{v}}^\sigma \;=\; \mathbf{0}\,, \\[2ex] K(\underline{\rho}^\sigma)\,\partial^2_{\xi\xi}\underline{\rho}^\sigma \;+\; \frac{1}{2}\,\partial_\xi K(\underline{\rho}^\sigma)\;\partial_\xi\underline{\rho}^\sigma \;-\; g_0(\underline{\rho}^\sigma) \;+\; g_0(\rho_\infty) \;-\; \frac{1}{2}\,(\underline{u}^\sigma - \sigma)^2 \;+\; \frac{1}{2}\,(u_\infty - \sigma)^2 \;=\; 0\,, \end{array}\right.$$

where $\underline{u}^\sigma := \underline{\mathbf{u}}^\sigma\cdot\mathbf{n}$ and $\underline{\mathbf{v}}^\sigma := \underline{\mathbf{u}}^\sigma - \underline{u}^\sigma\mathbf{n}$. Therefore, a dynamical solitary wave, for which $\underline{u}^\sigma \neq \sigma$, is such that $\underline{\mathbf{v}}^\sigma$ is constant and $(\underline{\rho}^\sigma, \underline{u}^\sigma)$ satisfy the one-dimensional profile equation (2.13). By change of

Galilean frame, we may assume without loss of generality that $\underline{\mathbf{v}}^\sigma$ is zero. Moreover, similarly as in one space dimension, the change of Galilean frame $(\mathbf{x},t)\mapsto(\mathbf{x}-\sigma t\mathbf{n},t)$ changes (1.2) into

$$\left\{\begin{array}{l}\partial_t\,\rho\,+\,\mathrm{div}\,(\rho\,(\mathbf{u}-\sigma\mathbf{n}))\,=\,0\,,\\[2ex] \partial_t\mathbf{u}\,+\,((\,\mathbf{u}-\sigma\mathbf{n})\cdot\nabla\,)\,\mathbf{u}\,+\,\nabla g_0\,=\,\nabla(\,K\,\Delta\rho\,+\,\frac{1}{2}\,K'_\rho\,|\nabla\rho|^2\,)\,,\end{array}\right. \tag{3.32}$$

of which $(\underline{\rho}^\sigma,\underline{\mathbf{u}}^\sigma)$ is a *stationary* solution. Linearizing (3.32) about $(\underline{\rho}^\sigma,\underline{\mathbf{u}}^\sigma)$ we get

$$\partial_t\dot{\mathbf{U}}\,=\,\mathbf{L}^\sigma\cdot\dot{\mathbf{U}}\,,\quad\text{with }\ \dot{\mathbf{U}}\,:=\,\begin{pmatrix}\dot\rho\\ \dot{\mathbf{u}}\end{pmatrix}\,,$$

$$\mathbf{L}^\sigma\cdot\dot{\mathbf{U}}\,:=\,\begin{pmatrix}-\,\mathrm{div}\Big((\underline{\mathbf{u}}^\sigma-\sigma\mathbf{n})\dot\rho\,+\,\underline{\rho}^\sigma(\dot{\mathbf{u}}-\sigma\mathbf{n})\Big)\\ -(\underline{u}^\sigma-\sigma)\,\partial_\xi\dot{\mathbf{u}}\,-\,(\dot u-\sigma)\partial_\xi\underline{u}^\sigma\,\mathbf{n}\,+\,\nabla\Big(-\underline{\alpha}^\sigma\,\dot\rho\,+\,\underline{K}^\sigma\,\Delta\dot\rho\,+\,\partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big)\end{pmatrix}\,,$$

where $\xi:=\mathbf{x}\cdot\mathbf{n}-\sigma t$, and, as in Section 2,

$$\underline{K}^\sigma\,:=\,K(\underline{\rho}^\sigma)\,,\quad\underline{\alpha}^\sigma\,:=\,\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\underline{\rho}^\sigma)\,-\,\frac{\mathrm{d}K}{\mathrm{d}\rho}(\underline{\rho}^\sigma)\,\partial^2_{\xi\xi}\underline{\rho}^\sigma\,-\,\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2}(\underline{\rho}^\sigma)\,(\partial_\xi\underline{\rho}^\sigma)^2\,.$$

A necessary condition for the linearized stability of $(\underline{\rho}^\sigma,\underline{\mathbf{u}}^\sigma)$ is that $\mathbf{L}^\sigma$ has no spectrum in the open right half-plane. Equivalently, the operator $L^\sigma(\eta)$, obtained by Fourier transform in the direction normal to $\mathbf{n}$, the corresponding wave vector being denoted by $\eta\in\mathbb{R}^{d-1}$, has no spectrum in the open right half-plane. To obtain the explicit form of $L^\sigma(\eta)$, we may assume without loss of generality - because of rotational invariance of (1.2) -, that $\mathbf{n}$ is the last vector e_d of the canonical basis in $\mathbb{R}^d$. Hence we may identify the vector $\dot{\mathbf{v}}\in\mathrm{e}_d^\perp$ with a vector in $\mathbb{R}^{d-1}=\mathrm{span}\;(\mathrm{e}_1,\dots,\mathrm{e}_{d-1})$, and $\dot{\mathbf{U}}$ and $L^\sigma(\eta)\cdot\dot{\mathbf{U}}$ with

$$\begin{pmatrix}\dot\rho\\ \dot{\mathbf{v}}\\ \dot u\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}-\,\partial_\xi\Big((\underline{u}^\sigma-\sigma)\dot\rho\,+\,\underline{\rho}^\sigma(\dot u-\sigma)\Big)\,-\,i\,\underline{\rho}^\sigma\eta\cdot\dot{\mathbf{v}}\\ -(\underline{u}^\sigma-\sigma)\,\partial_\xi\dot{\mathbf{v}}\,+\,i\Big(-(\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\,\dot\rho\,+\,\underline{K}^\sigma\,\partial^2_{\xi\xi}\dot\rho\,+\,\partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big)\,\eta\\ \partial_\xi\Big(-(\underline{u}^\sigma-\sigma)\,\dot u\,-\,(\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\,\dot\rho\,+\,\underline{K}^\sigma\,\partial^2_{\xi\xi}\dot\rho\,+\,\partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big)\end{pmatrix}$$

respectively. The operator $L^\sigma(\eta)$ is clearly similar to the real-valued operator

$$\widetilde{L}^\sigma(\eta):\begin{pmatrix}\dot\rho\\ \dot{\tilde{\mathbf{v}}}\\ \dot u\end{pmatrix}\mapsto\begin{pmatrix}-\,\partial_\xi\Big((\underline{u}^\sigma-\sigma)\dot\rho\,+\,\underline{\rho}^\sigma(\dot u-\sigma)\Big)\,-\,\underline{\rho}^\sigma\eta\cdot\dot{\tilde{\mathbf{v}}}\\ -(\underline{u}^\sigma-\sigma)\,\partial_\xi\dot{\tilde{\mathbf{v}}}\,-\,\Big(-(\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\,\dot\rho\,+\,\underline{K}^\sigma\,\partial^2_{\xi\xi}\dot\rho\,+\,\partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big)\,\eta\\ \partial_\xi\Big(-(\underline{u}^\sigma-\sigma)\,\dot u\,-\,(\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\,\dot\rho\,+\,\underline{K}^\sigma\,\partial^2_{\xi\xi}\dot\rho\,+\,\partial_\xi\underline{K}^\sigma\,\partial_\xi\dot\rho\Big)\end{pmatrix}$$

Therefore, the spectra of $L^\sigma(\eta)$ and $\widetilde{L}^\sigma(\eta)$ coincide. From now on, we concentrate on $\widetilde{L}^\sigma(\eta)$ and we omit the tildas for simplicity. The asymptotic operator at $\pm\infty$ is

$$L^\sigma_\infty(\eta):\begin{pmatrix}\dot\rho\\ \dot{\mathbf{v}}\\ \dot u\end{pmatrix}\mapsto\begin{pmatrix}-\,(u_\infty-\sigma)\,\partial_\xi\dot\rho\,-\,\rho_\infty\partial_\xi\dot u\,-\,\rho_\infty\eta\cdot\dot{\mathbf{v}}\\ -(u_\infty-\sigma)\,\partial_\xi\dot{\mathbf{v}}\,-\,\Big(-(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)+K(\rho_\infty)\|\eta\|^2)\,\dot\rho\,+\,K(\rho_\infty)\,\partial^2_{\xi\xi}\dot\rho\Big)\,\eta\\ -(u_\infty-\sigma)\partial_\xi\dot u\,-\,(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)+K(\rho_\infty)\|\eta\|^2)\,\partial_\xi\dot\rho\,+\,K(\rho_\infty)\,\partial^3_{\xi\xi\xi}\dot\rho\Big)\end{pmatrix}$$

By Fourier tranform in ξ, we find that $\tau\in\mathbb{C}$ belongs to the spectrum of $L^\sigma_\infty(\eta)$ if and only if there exists $\zeta\in\mathbb{R}$ so that, either $\tau=-i(u_\infty-\sigma)\zeta$, or

$$(\tau\,+\,i\,(u_\infty-\sigma)\zeta)^2\,+\,\rho_\infty\left(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)\,+\,K(\rho_\infty)\,(\|\eta\|^2+\zeta^2)\right)\,(\|\eta\|^2+\zeta^2)\,=\,0\,. \tag{3.33}$$

Therefore, in all cases, τ is purely imaginary. As for the one-dimensional operator L^σ studied in Section 2, this implies the essential spectrum of $L^\sigma(\eta)$ coincides with the imaginary axis. Consequently, the (neutral) linearized stability of $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ will be determined by the point spectrum of $L^\sigma(\eta)$. As for L^σ, possible unstable eigenvalues τ (with $\mathrm{Re}\tau > 0$) of $L^\sigma(\eta)$ can be characterized as zeroes of an Evans function $\tau \mapsto D(\tau;\eta)$. Viewed as a function of (τ,η), D can be made analytic along rays (as was pointed out by Zumbrun and Serre in [22] for second order operators associated with viscous shocks; also see [20]). Furthermore, since $L^\sigma(\eta)$ is real valued, D can be chosen to be real for real τ. Therefore, the comparison of the signs of $D(\lambda\tau;\lambda\eta)$ for λ close to zero and for large λ provides a sufficient condition for instability, by the mean value theorem argument usually valid only in one space dimension. Another way is the one pointed out in [22, Lemma 7.5], which goes as follows in our situation. By nature of the solitary wave there is a function P (which we shall compute explicitly), homogeneous of degree 2, such that $D(\lambda\tau;\lambda\eta) \sim \lambda^2 P(\tau;\eta)$ as λ goes to zero. It will turn out that for a one-d stable solitary wave, P vanishes at points of the form $(\tau_0(\eta),\eta)$. Observing that $p^{(\lambda,\eta)}(\tau) := \lambda^{-2} D(\lambda\tau;\lambda\eta)$ defines a family a holomorphic functions on $\{\mathrm{Re}\tau > 0\}$, depending continuously on $(\lambda,\eta) \in \mathbb{R}^+ \times \mathbb{R}^{d-1}$, Rouché's theorem will then imply the existence of a continuous branch $\tau_*(\lambda,\eta)$ close to $\tau_0(\eta)$ for λ close to 0 such that $p^{(\lambda,\eta)}(\tau_*(\lambda,\eta)) = 0$, hence

$$D(\tau_\sharp(\eta);\eta) = 0$$

with $\tau_\sharp(\eta) := \|\eta\|\,\tau_*(\|\eta\|, \eta/\|\eta\|)$.

3.2 The Evans function computations

We proceed similarly as in Section 2. (The following computation is also close to the one in [3] for heteroclinic planar traveling waves.) We first rewrite the eigenvalue equations $(L^\sigma(\eta) - \tau)\dot{\mathbf{U}} = 0$ as a first order system of ODEs,

$$(B^\sigma(\eta)\,\Phi)' = A^\sigma(\tau;\eta)\,\Phi\,, \tag{3.34}$$

$$\Phi := \begin{pmatrix} \dot\rho \\ \dot\rho' \\ \dot\rho'' \\ \dot{\mathbf{v}} \\ \dot u \end{pmatrix},\quad B^\sigma(\eta) := \begin{pmatrix} 1 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 1 & 0 & 0^*_{d-1} & 0 \\ -(\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2) & (\underline{K}^\sigma)' & \underline{K}^\sigma & 0^*_{d-1} & -(\underline{u}^\sigma - \sigma) \\ 0_{d-1} & 0_{d-1} & 0_{d-1} & (\underline{u}^\sigma - \sigma)\,\mathbf{I}_{d-1} & 0_{d-1} \\ (\underline{u}^\sigma - \sigma) & 0 & 0 & 0^*_{d-1} & \underline{\rho}^\sigma \end{pmatrix},$$

$$A^\sigma(\tau;\eta) := \begin{pmatrix} 0 & 1 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 1 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & 0^*_{d-1} & \tau \\ (\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\eta & -(\underline{K}^\sigma)'\eta & -\underline{K}^\sigma\eta & -\tau\mathbf{I}_{d-1} & 0_{d-1} \\ -\tau & 0 & 0 & -\underline{\rho}^\sigma\,\eta^{\mathrm{t}} & 0 \end{pmatrix}.$$

The eigenvalues of the asymptotic system

$$(B^\sigma_\infty(\eta)\,\Phi)' = A^\sigma_\infty(\tau;\eta)\,\Phi \tag{3.35}$$

are $\omega_0^\sigma(\tau) := -\tau/(u_\infty - \sigma)$ and the roots ω of the dispersion relation

$$(\tau + (u_\infty - \sigma)\omega)^2 + \left(c_\infty^2 + \rho_\infty K_\infty(\|\eta\|^2 - \omega^2)\right)(\|\eta\|^2 - \omega^2) = 0 \tag{3.36}$$

(obtained from (3.33) by substituting ω for $i\zeta$). We assume from now on that u_∞ is greater than σ, so that $\omega_0^\sigma(\tau)$ is negative for positive τ, and thus contributes to the stable manifold of (3.35). In addition, it is found to be of geometric multiplicity $d-1$, the associated eigenspace of $B^\sigma_\infty(\eta)^{-1}A^\sigma_\infty(\tau;\eta)$ being spanned by the vectors

$$\mathbf{W}_0^{j,\sigma}(\tau;\eta) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau\, \mathrm{e}_j \\ (u_\infty - \sigma)\eta_j \end{pmatrix},\ j \in \{1, \dots, d-1\}$$

for $(\tau, \eta) \neq (0,0)$. Since these vectors $\mathbf{W}_0^{j,\sigma}$ are homogeneous in (τ, η), we may renormalize them and assume that they are homogeneous degree 0, that is, constant along rays $\{(\lambda\tau, \lambda\eta)\,;\ \lambda > 0\}$. Like the simpler equation (2.17), Eq. (3.36) has no purely imaginary root when $\mathrm{Re}\tau$ is positive. Thus the number of roots of negative real parts is independent of (τ, η), within the half-space $\{\mathrm{Re}\tau > 0\}$. As already seen in the case $\eta = 0$ (in which (3.36) degenerates to (2.17)), this number is two. We denote by $\omega_1^\sigma(\tau;\eta)$ and $\omega_2^\sigma(\tau;\eta)$ those roots. In the same way we find that (3.36) has two roots of positive real parts, $\omega_3^\sigma(\tau;\eta)$ and $\omega_4^\sigma(\tau;\eta)$ say. (Observe that $\omega_j^\sigma(\tau;\eta)$ are distinct from $\omega_0^\sigma(\tau)$ for $\tau \neq (u_\infty - \sigma)\|\eta\|$.) We choose their numbering according to their behavior as λ goes to zero along the ray $\{(\lambda\tau, \lambda\eta)\,;\ \lambda > 0\}$. More precisely, we have

$$\omega_1^\sigma(\lambda\tau;\lambda\eta) \to -\sqrt{(c_\infty^2 - (u_\infty-\sigma)^2)/(\rho_\infty K_\infty)}\,, \quad \omega_4^\sigma(\lambda\tau;\lambda\eta) \to +\sqrt{(c_\infty^2 - (u_\infty-\sigma)^2)/(\rho_\infty K_\infty)}$$

$$\omega_2^\sigma(\lambda\tau;\lambda\eta) \sim \lambda\, \underline{\omega}_2^\sigma(\tau;\eta)\,, \quad \omega_3^\sigma(\lambda\tau;\lambda\eta) \sim \lambda\, \underline{\omega}_3^\sigma(\tau;\eta)\,,$$

as λ goes to zero, where $\underline{\omega}_{2,3}^\sigma(\tau;\eta)$ are the roots of

$$(\tau + (u_\infty - \sigma)\omega)^2 + c_\infty^2\, (\|\eta\|^2 - \omega^2) = 0\,. \tag{3.37}$$

By definition, $\mathrm{Re}\underline{\omega}_2^\sigma < 0$ and $\mathrm{Re}\underline{\omega}_3^\sigma > 0$. Associated eigenvectors of $B_\infty^\sigma(\eta)^{-1} A_\infty^\sigma(\tau;\eta)$ are

$$\mathbf{W}_j^\sigma(\tau;\eta) := \begin{pmatrix} \rho_\infty \\ \rho_\infty\, \omega_j^\sigma(\tau;\eta) \\ \rho_\infty\, \omega_j^\sigma(\tau;\eta)^2 \\ \frac{\tau + (u_\infty-\sigma)\omega_j^\sigma(\tau;\eta)}{\omega_j^\sigma(\tau;\eta)^2 - \|\eta\|^2}\,\eta \\ -\omega_j^\sigma(\tau;\eta)\, \frac{\tau + (u_\infty-\sigma)\omega_j^\sigma(\tau;\eta)}{\omega_j^\sigma(\tau;\eta)^2 - \|\eta\|^2} \end{pmatrix}. \tag{3.38}$$

With this choice we have

$$\lim_{\lambda \searrow 0} \mathbf{W}_{1,4}^\sigma(\lambda\tau;\lambda\eta) = \begin{pmatrix} \rho_\infty \\ \rho_\infty\, \omega_{1,4}^\sigma(0;0) \\ \rho_\infty\, \omega_{1,4}^\sigma(0;0)^2 \\ 0_{d-1} \\ -(u_\infty - \sigma) \end{pmatrix},$$

$$\lim_{\lambda \searrow 0} \mathbf{W}_{2,3}^\sigma(\lambda\tau;\lambda\eta) = \begin{pmatrix} \rho_\infty \\ 0 \\ 0 \\ \dfrac{\tau + (u_\infty-\sigma)\underline{\omega}_{2,3}^\sigma(\tau;\eta)}{\underline{\omega}_{2,3}^\sigma(\tau;\eta)^2 - \|\eta\|^2}\,\eta \\ -\underline{\omega}_{2,3}^\sigma(\tau;\eta)\, \dfrac{\tau + (u_\infty-\sigma)\underline{\omega}_{2,3}^\sigma(\tau;\eta)}{\underline{\omega}_{2,3}^\sigma(\tau;\eta)^2 - \|\eta\|^2} \end{pmatrix} = \begin{pmatrix} \rho_\infty \\ 0 \\ 0 \\ \dfrac{c_\infty^2}{\tau + (u_\infty-\sigma)\underline{\omega}_{2,3}^\sigma(\tau;\eta)}\,\eta \\ -\dfrac{c_\infty^2\, \underline{\omega}_{2,3}^\sigma(\tau;\eta)}{\tau + (u_\infty-\sigma)\underline{\omega}_{2,3}^\sigma(\tau;\eta)} \end{pmatrix}.$$

By the method of Zumbrun *et al* [20, 22], we can construct an Evans function D^σ, analytic along rays $\{(\lambda\tau, \lambda\eta)\,;\ \lambda > 0\}$ and real valued for $\tau \in [0, +\infty)$, such that

$$D^\sigma(\tau;\eta) = 0\,,\ \mathrm{Re}\tau > 0 \iff \mathrm{Ker}\,(L^\sigma(\eta) - \tau) \neq \{0\}\,.$$

By definition, away from glancing points,

$$D^\sigma(\tau;\eta) \;=\; \det(\Phi_0^{1,\sigma}(\tau;\eta),\ldots,\Phi_0^{d-1,\sigma}(\tau;\eta),\Phi_1^\sigma(\tau;\eta),\Phi_2^\sigma(\tau;\eta),\Phi_3^\sigma(\tau;\eta),\Phi_4^\sigma(\tau;\eta))|_{\xi=0}\,,$$

where $\Phi_j(\tau;\eta)$ are solutions of (3.34) such that

$$\begin{aligned}&\Phi_0^{j,\sigma}(\tau;\eta) \;\overset{\xi\to+\infty}{\sim}\; \mathrm{e}^{\omega_0^\sigma(\tau;\eta)\xi}\,\mathbf{W}_0^{j,\sigma}(\tau;\eta)\,,\quad \Phi_{0,1,2}(\tau;\eta) \;\overset{\xi\to+\infty}{\sim}\; \mathrm{e}^{\omega_{0,1,2}(\tau;\eta)\xi}\,\mathbf{W}_{0,1,2}^\sigma(\tau;\eta)\,,\\ &\Phi_{3,4}(\tau;\eta) \;\overset{\xi\to-\infty}{\sim}\; \mathrm{e}^{\omega_{3,4}^\sigma(\tau;\eta)\xi}\,\mathbf{W}_{3,4}^\sigma(\tau;\eta)\,.\end{aligned} \tag{3.39}$$

Since $L^\sigma(0)\cdot(\underline{\mathbf{U}}^\sigma)' \;=\; 0$ and $(\underline{\mathbf{U}}^\sigma)'$ goes exponentially fast to zero at $\pm\infty$, as in dimension 1 we observe that

$$D^\sigma(\tau;\eta) \;=\; -\,r^2\,\det(\Phi_0^{1,\sigma}(\tau;\eta),\ldots,\Phi_0^{d-1,\sigma}(\tau;\eta),\breve\Phi_1^\sigma(\tau;\eta),\Phi_2^\sigma(\tau;\eta),\Phi_3^\sigma(\tau;\eta),\breve\Phi_4^\sigma(\tau;\eta))|_{\xi=0}\,,$$

where

$$\breve\Phi_1^\sigma(0;0) \;=\; \breve\Phi_4^\sigma(0;0) \;=\; \begin{pmatrix}(\underline{\rho}^\sigma)'\\(\underline{\rho}^\sigma)''\\(\underline{\rho}^\sigma)'''\\0_{d-1}\\(\underline{u}^\sigma)'\end{pmatrix}. \tag{3.40}$$

For simplicity, we shall omit the $\breve{}$ hats in what follows. Eq. (3.40) obviously implies that $D^\sigma(0;0)=0$. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}D^\sigma(\lambda\tau;\lambda\eta)|_{\lambda=0}=0\,. \tag{3.41}$$

To prove this, we introduce notations for the components of Φ_j^σ and $\Psi_j^\sigma := \partial_\lambda\Phi_j^\sigma(\lambda\tau;\lambda\eta)$, namely,

$$\Phi_j^\sigma \;=\; \begin{pmatrix}\phi_j^\sigma\\(\phi_j^\sigma)'\\(\phi_j^\sigma)''\\\nu_j^\sigma\\\mu_j^\sigma\end{pmatrix},\quad \Psi_j^\sigma \;=\; \begin{pmatrix}\psi_j^\sigma\\(\psi_j^\sigma)'\\(\psi_j^\sigma)''\\\zeta_j^\sigma\\\chi_j^\sigma\end{pmatrix}.$$

By differentiation of $(B^\sigma(\lambda\eta)\,\Phi_j^\sigma(\lambda\tau;\lambda\eta))' \;=\; A^\sigma(\lambda\tau;\lambda\eta)\,\Phi_j^\sigma(\lambda\tau;\lambda\eta)$ with respect to λ, we obtain

$$(B^\sigma(0)\,\Psi_j^\sigma(0;0))' \;=\; A^\sigma(0;0)\,\Psi_j^\sigma(0;0)\;+\;A_1^\sigma(\tau;\eta)\,\Phi_j^\sigma(0;0)\,, \tag{3.42}$$

$$A_1^\sigma(\tau;\eta) \;:=\; \frac{\mathrm{d}}{\mathrm{d}\lambda}A^\sigma(\lambda\tau;\lambda\eta)|_{\lambda=0} \;=\; \begin{pmatrix}0&0&0&0_{d-1}^*&0\\0&0&0&0_{d-1}^*&0\\0&0&0&0_{d-1}^*&\tau\\(\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\eta&-(\underline{K}^\sigma)'\eta&-\underline{K}^\sigma\eta&-\tau\mathbf{I}_{d-1}&0_{d-1}\\-\tau&0&0&-\underline{\rho}^\sigma\,\eta^{\mathrm{t}}&0\end{pmatrix}.$$

By (3.40), we have

$$A_1^\sigma(\tau;\eta)\,\Phi_{1,4}^\sigma(0;0) \;=\; \begin{pmatrix}0\\0\\\tau(\underline{u}^\sigma)'\\((\underline{\alpha}^\sigma\,(\underline{\rho}^\sigma)'-(\underline{K}^\sigma)'(\underline{\rho}^\sigma)''-\underline{K}^\sigma(\underline{\rho}^\sigma)''')\,\eta\\-\tau(\underline{\rho}^\sigma)'\end{pmatrix}.$$

We thus see that the third row, respectively the last row, in (3.42) for $j = 1, 4$ are equivalent to the second (and last) row, respectively the first row, in

$$L^\sigma \cdot \begin{pmatrix} \psi^\sigma_{1,4}(0;0) \\ \chi^\sigma_{1,4}(0;0) \end{pmatrix} = \tau \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix} = -\tau \, L^\sigma \cdot \begin{pmatrix} \partial_\sigma \underline{\rho}^\sigma \\ \partial_\sigma \underline{u}^\sigma \end{pmatrix},$$

where L^σ is the one-dimensional operator of Section 2. Therefore, up to adding a constant times $\lambda \Phi^\sigma_{1,4}(\lambda\tau;\lambda\eta)$ to $\Phi^\sigma_{1,4}(\lambda\tau;\lambda\eta)$, we may assume that

$$\begin{pmatrix} \psi^\sigma_1(0;0) \\ \chi^\sigma_1(0;0) \end{pmatrix} = \begin{pmatrix} \psi^\sigma_4(0;0) \\ \chi^\sigma_4(0;0) \end{pmatrix} = -\tau \begin{pmatrix} \partial_\sigma(\underline{\rho}^\sigma) \\ \partial_\sigma(\underline{u}^\sigma) \end{pmatrix}. \tag{3.43}$$

Now the intermediate $(d-1)$ rows in (3.42) for $j = 1, 4$ read

$$((\underline{u}^\sigma - \sigma)\,\zeta_{1,4}(0;0))' = ((\underline{\alpha}^\sigma\,(\underline{\rho}^\sigma)' - (\underline{K}^\sigma)'(\underline{\rho}^\sigma)'' - \underline{K}^\sigma\,(\underline{\rho}^\sigma)''')\,\eta = -\Big(\frac{1}{2}(\underline{u}^\sigma - \sigma)^2\Big)'\,\eta$$

by the profile equation (2.13). Therefore, by integration,

$$\zeta_1(0;0) = \zeta_4(0;0) = -\frac{1}{2}\Big((\underline{u}^\sigma - \sigma) - \frac{(u_\infty - \sigma)^2}{\underline{u}^\sigma - \sigma}\Big)\,\eta\,.$$

So finally, we have

$$\Psi^\sigma_1(0;0) = \Psi^\sigma_4(0;0)\,, \tag{3.44}$$

which together with (3.40) implies (3.41), and

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} D^\sigma(\lambda\tau;\lambda\eta)|_{\lambda=0} = \det(\Phi^{1,\sigma}_0, \dots, \Phi^{d-1,\sigma}_0, \Phi^\sigma_1, \Phi^\sigma_2, \Phi^\sigma_3, \partial^2_{\lambda\lambda}(\Phi^\sigma_4 - \Phi^\sigma_1))(\lambda\tau;\lambda\eta)|_{\lambda=0}|_{\xi=0}\,.$$

For simplicity, in what follows we omit the superscript σ, and we just denote Φ_j for $\Phi^\sigma_j(0;0)$, and Θ_j for $\partial^2_{\lambda\lambda}\Phi^\sigma_j(\lambda\tau;\lambda\eta)|_{\lambda=0}$. The starting point is to evaluate the determinant above is to note that

$$\det B = \underline{\rho}\,\underline{K}\,(\underline{u} - \sigma)^{d-1} \neq 0\,,$$

hence

$$\begin{aligned} &\det(\Phi^1_0, \dots, \Phi^{d-1}_0, \Phi_1, \Phi_2, \Phi_3, \partial^2_{\lambda\lambda}(\Phi^\sigma_4 - \Phi^\sigma_1)) \\ &= \frac{1}{\underline{\rho}^\sigma\,\underline{K}^\sigma(\underline{u} - \sigma)^{d-1}}\ \det(B\Phi^1_0, \dots, B\Phi^{d-1}_0, B\Phi_1, B\Phi_2, B\Phi_3, \partial^2_{\lambda\lambda}B(\Phi^\sigma_4 - \Phi^\sigma_1))\,. \end{aligned}$$

By construction of Φ_j, since all but the first two rows of $A(0;0)$ are zero, we have

$$B(0)\,\Phi_j = \begin{pmatrix} \phi_j \\ \phi'_j \\ R_j \end{pmatrix},$$

where R_j is a *constant vector* in $\mathbb{R}^{d+1}$ determined by the asymptotic behovior of Φ_j. In particular R_1 is the null vector. We shall compute the other vectors R_j later on. We also need some information on $S_{1,4} : \xi \to S_{1,4}(\xi) \in \mathbb{R}^2$ such that, by definition,

$$B\,\Theta_{1,4} = \begin{pmatrix} \theta_{1,4} \\ \theta'_{1,4} \\ S_{1,4} \end{pmatrix}.$$

Differentiating twice $(B(\lambda\eta)\,\Phi_j(\lambda\tau;\lambda\eta))' \,=\, A(\lambda\tau;\lambda\eta)\,\Phi_j(\lambda\tau;\lambda\eta)$ with respect to λ, we obtain

$$(B(0)\,\Theta_j \,+\, B_2(\eta)\,\Phi_j)' \,=\, A(0;0)\,\Theta_j \,+\, 2\,A_1^\sigma(\tau;\eta)\,\Psi_j\,, \tag{3.45}$$

$$B_2(\eta) \,:=\, \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}B(\lambda\eta)|_{\lambda=0} \,=\, \begin{pmatrix} 0 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & 0^*_{d-1} & 0 \\ -2\underline{K}\|\eta\|^2 & 0 & 0 & -\tau\mathbf{0}_{d-1} & 0_{d-1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, we have by (3.43) and (3.43),

$$S'_{1,4} \,=\, 2\begin{pmatrix} -\tau^2\partial_\sigma\underline{u} \,+\, (\underline{K}\,\varrho')'\|\eta\|^2 \\ -\tau(\underline{\alpha}\partial_\sigma\varrho - \underline{K}'\partial_\sigma\varrho' - \underline{K}\partial_\sigma\varrho'')\eta \,+\, \frac{1}{2}\tau\Big((\underline{u}^\sigma - \sigma) \,-\, \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\Big)\,\eta \\ \tau^2\partial_\sigma\varrho \,+\, \frac{1}{2}\varrho\Big((\underline{u}-\sigma) \,-\, \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\Big)\,\|\eta\|^2 \end{pmatrix}.$$

Lemma 2 *If* Π *denotes the projection operator*

$$\Pi : \begin{pmatrix} \phi \\ \phi' \\ R \end{pmatrix} \in \mathbb{C}^{d+3} \mapsto R \in \mathbb{C}^{d+1}\,,$$

then, if $\tau^2 \,\neq\, (u_\infty-\sigma)^2\,\|\eta\|^2$*, the vectors* $R_0^j := \Pi B(0)\Phi_0^j$*,* $j = 1,\dots,d-1$*, and* $R_k := \Pi B(0)\Phi_k$*,* $k = 2,3$ *are independent.*

Remark 3 *Points* (τ,η) *with* $\tau^2 \,=\, (u_\infty-\sigma)^2\,\|\eta\|^2$ *are glancing points, for which* ω_2 *coincides with* ω_0*. Our computation below does not imply at all that the second order order derivative of the Evans function vanishes at those points: a different computation should be made to find the actual value of that derivative.*

Proof. [Lemma 2] We easily compute that

$$R_0^j \,=\, (u_\infty-\sigma)\begin{pmatrix} -(u_\infty-\sigma)\,\eta_j \\ \tau\mathrm{e}_j \\ \rho_\infty\,\eta_j \end{pmatrix},$$

and for $k = 2,3$,

$$R_k \,=\, \frac{1}{\tau + (u_\infty-\sigma)\underline{\omega}_k}\begin{pmatrix} -c_\infty^2\,\tau \\ (u_\infty-\sigma)c_\infty^2\,\eta \\ \rho_\infty\,((u_\infty-\sigma)\,(\tau+(u_\infty-\sigma)\underline{\omega}_k) \,-\, c_\infty^2\,\omega_k) \end{pmatrix},$$

hence

$$\det(R_0^1,\dots,R_0^{d-1},R_2,R_3) \,=\, c_\infty^2\,(\omega_2-\omega_3)\,(c_\infty^2-(u_\infty-\sigma)^2)\,(\tau^2-(u_\infty-\sigma)^2\|\eta\|^2)\,.$$

Thanks to Lemma 2, we may proceed as in Section 2. We introduce (the unique) numbers d_0^j, $j = 1,\dots,d-1$, and $d_{2,3}$ such that

$$S_4 - S_1 \,=\, \sum_{j=1}^{d-1} d_0^j R_0^j \,+\, d_2\,R_2 \,-\, d_3\,R_3\,,$$

and develop the determinant as follows,

$$\det(B(0)\Phi_0^1,\dots,B(0)\Phi_0^{d-1},B\Phi_1,B(0)\Phi_2,B(0)\Phi_3,\partial^2_{\lambda\lambda}B(0)(\Phi_4-\Phi_1)) \;=$$

$$\left|\begin{array}{ccccccc} \phi_0^1 & \dots & \phi_0^{d-1} & \underline{\rho}' & \phi_2 & \phi_3 & \widetilde{\theta}_4-\widetilde{\theta}_1 \\ (\phi_0^1)' & \dots & (\phi_0^{d-1})' & \underline{\rho}'' & \phi_2' & \phi_3' & \widetilde{\theta}_4'-\widetilde{\theta}_1' \\ R_0^1 & \dots & R_0^{d-1} & 0_{d+1} & R_2 & R_3 & 0_{d+1} \end{array}\right| =$$

$$\det(R_0^1,\dots,R_0^{d-1},R_2,R_3)\left|\begin{array}{cc} \underline{\rho}' & \widetilde{\theta}_4-\widetilde{\theta}_1 \\ \underline{\rho}'' & \widetilde{\theta}_4'-\widetilde{\theta}_1' \end{array}\right|$$

with

$$\widetilde{\theta}_4 \;:=\; \theta_4 \,+\, d_3\,\phi_3\,,\quad \widetilde{\theta}_1 \;:=\; \theta_1 \,+\, \sum_{j=1}^{d-1} d_0^j\,\phi_0^j \,+\, d_2\,\phi_2\,.$$

By the same technique as in Section 2 we find that

$$\left|\begin{array}{cc} \underline{\rho}' & \widetilde{\theta}_4-\widetilde{\theta}_1 \\ \underline{\rho}'' & \widetilde{\theta}_4'-\widetilde{\theta}_1' \end{array}\right|_{|\xi=0} \;=\; \frac{1}{\underline{K}(0)}\int_{-\infty}^{+\infty} s[\widetilde{\theta}_{1,4}]\,\underline{\rho}'\,,$$

with

$$s[\widetilde{\theta}_4] \;:=\; \underline{K}\,\widetilde{\theta}_4'' \,+\, (\underline{K})'\,\widetilde{\theta}_4' \,-\, \underline{\alpha}\,\widetilde{\theta}_4 \,+\, \frac{1}{\underline{\rho}}(\underline{u}-\sigma)^2\,\widetilde{\theta}_4 \;=\; (1,0^*_{d-1},(\underline{u}-\sigma)/\underline{\rho})\,(S_4\,+\,d_3\,R_3)$$

$$=\; (1,0^*_{d-1},(\underline{u}-\sigma)/\underline{\rho})\,(S_1\,+\,\sum_{j=1}^{d-1} d_0^j\,R_0^j\,+\,d_2\,R_2) \;=:\; s[\widetilde{\theta}_1]\,.$$

Since

$$(\underline{u}-\sigma)\,\underline{\rho}' \;=\; -\underline{\rho}\,\underline{u}'\,,$$

we have

$$\int_{-\infty}^{+\infty} s[\widetilde{\theta}_{1,4}]\,\underline{\rho}' \;=\; \int_{-\infty}^{+\infty}(\underline{\rho}',0^*_{d-1},-\underline{u}')\,S_4 \;=\; 2\int_{-\infty}^{+\infty}\tau^2\Big((\underline{\rho}-\rho_\infty)\,\partial_\sigma\underline{u}\,+\,(\underline{u}-u_\infty)\,\partial_\sigma\underline{\rho}\Big)$$

$$+\int_{-\infty}^{+\infty}\|\eta\|^2\Big(2\,\underline{K}\,(\underline{\rho}')^2\,+\,\underline{\rho}\,(\underline{u}-u_\infty)\,\big((\underline{u}-\sigma)\,-\,\frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\big)\Big)$$

after integration by part. In factor of τ^2 we recognize $-m''(\sigma)$ (see (2.10)), and the factor of $\|\eta\|^2$ is obviously positive, since

$$2\,\underline{K}\,(\underline{\rho}')^2+\underline{\rho}\,(\underline{u}-u_\infty)\,\big((\underline{u}-\sigma)-\frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\big)\;\geq\;\frac{\underline{\rho}}{\underline{u}-\sigma}\,(\underline{u}-\sigma-(u_\infty-\sigma))^2\,(\underline{u}-\sigma+u_\infty-\sigma)\;>\;0\,.$$

(Recall that as $j=\underline{\rho}\,(\underline{u}-\sigma)\;=\;\rho_\infty(u_\infty-\sigma)$ has been assumed positive.) In conclusion, if $(\tau;\eta)$ is not a glancing point, for λ close to 0, we have $D(\lambda\tau,\lambda\eta)\sim\lambda^2\,P(\tau;\eta)$ with

$$P(\tau;\eta)\;=\;-r^2\,(-m''(\sigma)\,\tau^2\,+\,s^2\,\|\eta\|^2)\,,$$

where r and s are nonzero real numbers. If $m''(\sigma)<0$, which implies that the solitary wave is one-d unstable by Theorem 1, perturbations transverse to the wave makes the local behavior of the Evans function even ‘worse’. If $m''(\sigma)>0$, which implies that the solitary wave is orbitally stable in one space dimension, we find as announced above a continuous branch $\eta\mapsto\tau_\sharp(\eta)$ along which D vanishes. We have thus proved the following.

Theorem 2 *Planar solitary waves satisfying the one-dimensional stability condition* (2.25) *are linearly unstable in several space dimensions.*

In view of the method developed recently by Rousset and Tzvetkov [18], we expect that this *linear* transverse instability implies *nonlinear instability*. This will be the purpose of a separate paper.

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